Consequences

If \( A^2 = 1 \) it is always possible to find a basis in which the Hamiltonian is real

\[
[H, A] = 0
\]
\[
A^2 = 1
\]

Take a basis \( \phi_1, \phi_2, \ldots \)

\[
\psi_1 = \frac{1}{\sqrt{2}} (\phi_1 + A\phi_2)
\]

\[
\phi_2' = \phi_2 - (\phi_2, \psi_1)\psi_1 = (\phi_2, \psi_1) = 0
\]

\[
\psi_2 = \frac{1}{\sqrt{2}} (\phi_2' + A\phi_2')
\]

\[
= (\psi, \psi_2) = \frac{1}{\sqrt{2}} (\psi, \psi_1') + \frac{1}{\sqrt{2}} (\psi, A\phi_2')
\]

\[
= \frac{1}{\sqrt{2}} (\phi_1'|A\psi_1')
\]

Now look at \( H \) in the \( \psi_2 \) basis

\[
\langle \psi_1 | H | \psi_2 \rangle = \langle A\phi_1 | H + IA\phi_2' \rangle^*
\]

\[
= \langle A\phi_1 | H | \psi_2 \rangle^*
\]

\[
= \langle \psi_2 | H | \psi_2 \rangle
\]

\[
\Rightarrow H\psi_2 = \psi_2
\]

\[
H^* = H \Rightarrow H\psi_2 \in \mathbb{R}
\]

\[
\langle \psi_1 | A\phi_1'| \psi_2 \rangle = \langle A\phi_1 | A\phi_1'| \psi_2 \rangle^*
\]

\[
= \langle \phi_1 | \phi_1' \rangle^* = 0
\]