phase space, but, on the contrary, explores the whole energy surface (except for a set of zero measure\(^{(\star)}\)). In fact, an even stronger property holds for ergodic systems: one can show that almost every orbit explores almost every point of the energy surface \(S_E\) (densely covers \(S_E\)), spending in any part \(A\) a time proportional to the area of \(A\). Indeed, let \(\alpha \in S_E\) and \(t(\alpha,A,\varepsilon)\) be the time that \(\alpha\) spends in \(A \subset S_E\) between instants 0 and \(\varepsilon\). According to the ergodic theorem (Birkhoff-von Neumann), the limit
\[
\lim_{\varepsilon \to 0} \frac{t(\alpha,A,\varepsilon)}{\varepsilon} = \sigma(A)
\]
does exist for almost all \(\alpha\)'s, and is equal to the area of \(A\):

\[
\lim_{\varepsilon \to 0} \frac{t(\alpha,A,\varepsilon)}{\varepsilon} = \sigma(A)
\]
if the system is ergodic.

The equality of phase averages and time averages constitutes an alternative definition of ergodicity (i.e. is equivalent to metrical transitivity); it can be formulated as follows: a system is ergodic iff, for any integrable function \(f(\alpha)\) \((\alpha \in X, \int f(\alpha) \, d\sigma(\alpha))\),

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t dt \quad f(T_t \alpha) = \int_{S_E} f(\alpha) \, d\sigma
\]

for almost all \(\alpha \in S_E\). Eq.(III-5) is often referred to as a version of the law of large numbers, the left hand side representing the infinitely many trials approaching the probability in the right hand side.

Equilibrium Statistical Mechanics is built on the ergodic hypothesis. But ergodicity is not sufficient for a system to reach an equilibrium state: one additional property—called mixing—is needed, which concerns the way any volume element evolves with time.

\(\star\) It is clear that no trajectory can explore the whole energy surface (i.e. ergodicity in the sense of Boltzmann can never hold). Indeed, a trajectory which would pass through any point of \(S_E\) should intersect with itself, which is impossible. What we call ergodicity here is often referred to as the quasi-ergodic hypothesis in Statistical Mechanics.