The study of billiards is of interest in many respects:

i) they are, among the physical Hamiltonian systems, the most studied from a mathematical point of view, and some important analytical properties are known

ii) billiards are very rich in their behaviours: there exist systems lying at the top, as well as at the bottom of the hierarchy of regularity described in Sect. III.1, and also systems which cannot be classified according to this hierarchy, i.e. for which the phase space contains both regular and chaotic regions

iii) the energy surface of billiards is of dimension two, which is the smallest one needed to observe chaotic motion

iv) for our purpose–trying to find signatures of chaos in quantum systems whose classical analogues are chaotic by statistical studies of the spectrum–, drums (quantum billiards) are the most convenient systems: they have an infinite discrete spectrum, and efficient numerical methods are available to compute long series of adjacent levels.

Regular billiards

The rectangle, the circle, and the ellipse are regular billiards, i.e. dynamical systems whose energy surface is a torus. The two independent constants of motion are

i) for the rectangle: the two projections \(|v_x|\) and \(|v_y|\) of the moduli of the velocity on two axes parallel to the sides of the rectangle

ii) for the circle: the energy, and the angular momentum \(L_z = |\hat{\mathbf{L}}|\) with respect to the center of the billiard

iii) for the ellipse: the energy, and the product \(L_1L_2\) of the angular momenta with respect to the two foci

Fig. III.19 – Trajectories in regular billiards. Caustics can be seen for the circular billiard (the caustics are circles of same center as the billiard) and for the elliptic billiard (the caustics are ellipses and hyperbolae confocal with the ellipse defining the boundary of the billiard)

For the circle and the ellipse, the integrability manifests itself by the existence of families of caustics, which reveal the existence of tori (see again Ref. [Be-81 a], and also Ref.[Be-83]).