the characteristic property of ergodicity (III-5), translated for our discrete time process as:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{p=0}^{n-1} f(T^p \mathbf{x}) = \int_0^1 f(x) dx \quad \text{a.e.,} \quad (III-22)
\]

where \( f(x) \) is integrable on \([0,1]\) for the Lebesgue measure \( dx \). Choose now \( f \) to be the indicator \( I_{\Omega} \) (or characteristic function) of any subset \( \Omega \) of \([0,1]\) (of non-zero measure):

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{p=0}^{n-1} I_{T^p \mathbf{x}}(\mathbf{x}) = \int_{\Omega} f(x) dx = L(\Omega) \quad \text{a.e.,} \quad (III-23)
\]

where \( L(\Omega) \) is the length (the Lebesgue measure) of \( \Omega \). This equation simply tells us that the proportion of numbers generated by successive applications of \( T \) which belong to any interval \( \Omega \subset [0,1] \) asymptotically tends to the length \( L(\Omega) \) of \( \Omega \). Notice that this property is shared by all dynamical systems which are ergodic on \([0,1]\) for the Lebesgue measure, and in particular holds for the example i) (rotations of the circle): all these systems are such that the numbers generated by \( \{ T^m : m = 1,2,\ldots \} \) asymptotically have a uniform distribution on \([0,1]\). However, all these systems cannot be used as random number generators. Consider for example the rotations of the circle, which, as we have seen, produce points in an inexorable regular pattern: clearly, in this case, the successive points obtained by the mechanism \( T \) have the strongest correlations that can be imagined. On the contrary, the algorithm (III-18) produces completely uncorrelated points, or, in other words, the successive trials associated with successive applications of \( T \) are, in this case, completely independent.

Several consequences of the property just discussed are worth to be mentioned.

i) consider the case where the subset \( \Omega \) is any of the segments \( \Omega_k = [\frac{k}{r}, \frac{k+1}{r}] \) \((k = 0,1,2,\ldots, r-1)\). Then, \( I_{\Omega_k} (T^p \mathbf{x}) \) is equal to one if the \( p \)th digit of \( \mathbf{x} \) is \( k \), and zero otherwise. Thus, the repeated experiments \( \{ T^p \mathbf{x} : \mathbf{x} = 01,\ldots, n-1 \} \) can be viewed as independent trials for a random variable which takes the value \( k \) \((k = 0,\ldots, r-1)\) with probability \( \frac{1}{r} \). One can therefore interpret Eq.(III-23) as follows. The l.h.s. represents the relative frequency of the value \( k \), and the r.h.s. the probability \( \frac{1}{r} \) for getting \( k \). According to Eq.(III-23), the probability that the l.h.s. is equal to the r.h.s. is one, i.e. the relative frequency is almost surely equal to the probability, which is the strong law of large numbers (i.e. except for a set of starting points of zero measure).

ii) Another illustration of the strong law of large numbers is due to Borel, and can be recovered very nicely directly from the ergodic theorem. A number \( \mathbf{x} \in [0,1] \) is said to be normal to base \( r \) if each digit of its expansion in this basis has the relative frequency \( \frac{1}{r} \). A number is said to be normal if it is normal to every basis \( r \). As a consequence