Regular systems

Roughly speaking, a regular dynamical system is such that it can be integrated by quadratures. Let us be more precise, and define what are the conditions and the consequences of "extreme regularity". For a more rigorous account of the subject, see Refs. [Ar-76, AA-67].

We consider a time-independent Hamiltonian system with \( N \) degrees of freedom:

\[
H = H(q, p) \quad , \quad q = \{ q_1, \ldots, q_N \} \\
p = \{ p_1, \ldots, p_N \}
\]

The equations of motion, written in Hamiltonian form, are:

\[
\begin{align*}
\dot{q} &= \nabla_p H(q, p) \\
\dot{p} &= -\nabla_q H(q, p)
\end{align*}
\]

Definition

A time-independent Hamiltonian system with \( N \) degrees of freedom is said to be "integrable" if there exist \( N \) constants of motion \( F_1(q, p), \ldots, F_N(q, p) \) (one of them being \( H \) itself) which are analytic functions of \( q \) and \( p \), single valued, functionally independent, and in involution (for the Poisson bracket):

\[
\{ F_n, F_m \} = 0 \quad \forall \; n, m = 1, \ldots, N.
\]

Theorem (Liouville-Arnold)

If a system is integrable, then

i) there exist a canonical transformation to action-angle variables:

\[
(q_1, \ldots, q_N; p_1, \ldots, p_N) \rightarrow (\theta_1, \ldots, \theta_N; I_1, \ldots, I_N)
\]
such that the Hamiltonian, expressed in the new variables, depends only on the actions:

\[
H(q_1, \ldots, q_N; p_1, \ldots, p_N) \rightarrow \tilde{H}(I_1, \ldots, I_N) = \tilde{H}(I)
\]

The action variables are constants of motion:

\[
\dot{I} = 0
\]