Lecture Notes for Thermal Physics and Statistical Mechanics, Chapter 7

These lecture notes are based on the text book “Thermal Physics” by Kittel and Kroemer, which is the required textbook for this course.
7(a) Fermi-gas

Quantum concentration \( n = \left(\frac{8\pi T}{e^2}N\right)^{3/2} \)

\( n < n_0 \): classical gas

\( n \approx n \): degenerate Fermi gas

this regime is reached for \( T \to 0 \)

For \( T \to 0 \), all states up to the Fermi level are filled

\[ E_F \]

7(b) Ground state at zero temperature

\[ \varepsilon_n = \frac{kT}{2\eta} \left( \frac{\pi}{L} \right)^2 \left( n_L^2 + n_{\eta}^2 + n_{z}^2 \right) \quad n_{i} > 0 \]

\[ \eta_F = \frac{3W}{\pi} \]

\[ N = \frac{2c}{h^2} \int_{B = 0}^{1} \eta_F^3 \]

\[ \eta_F = \left( \frac{3W}{\pi} \right)^{1/3} \]

\[ \varepsilon_F = \frac{kT}{2\eta} \left( \frac{\pi}{L} \right)^2 \eta_F^2 = \frac{kT}{2\eta} \left( \frac{\pi}{L} \right)^2 \left( \frac{3W}{\pi} \right)^{2/3} \]

\[ = \frac{h^2}{2M} \left( \frac{3W^2}{\pi} \right)^{1/3} \]

\[ U_0 = \frac{1}{\hbar^2} \sum_{\eta \neq \eta_F} \varepsilon_\eta = \frac{1}{\hbar^2} \frac{\pi^2}{3} \int \eta_F^2 d\eta \]

\[ = \frac{\pi^2}{2M} \frac{h^2}{L^2} \int \eta_F^2 d\eta \]
\[ U_0 = \frac{\pi^2}{16m} \frac{t^2}{L^2} n^5 = \frac{2}{5} N \Sigma P \]

**Density of States**

\[ N(\varepsilon) \quad \text{total number of states below } \varepsilon \]

\[ \rho(\varepsilon) \quad \text{density of states} \]

\[ \rho(\varepsilon) = \frac{\text{# states in } (\varepsilon, \varepsilon + \Delta\varepsilon)}{\Delta\varepsilon} \]

\[ = \frac{dN(\varepsilon)}{d\varepsilon} \]

\[ N(\varepsilon) = \frac{\pi}{4} n^3 \varepsilon = \frac{\pi}{3} \left( \frac{e \hbar}{\varepsilon} \right)^2 (\varepsilon / \hbar)^{3/2} = \frac{V}{3} \pi^2 \left( \frac{e \hbar}{\varepsilon} \right)^{3/2} \varepsilon^{3/2} \]

\[ \rho(\varepsilon) = \frac{d(N(\varepsilon))}{d\varepsilon} = \frac{V}{e^2 \hbar^2} \left( \frac{e \hbar}{\varepsilon} \right)^{3/2} \]

In general

\[ U = \int_0^\infty \varepsilon \rho(\varepsilon) f(\varepsilon, t, u) d\varepsilon \]

\[ u = \int_0^\infty \varepsilon \rho(\varepsilon) f(\varepsilon, t, u) d\varepsilon \]

\[ \text{distribution function} \quad \text{eq. for distribution} \]
Heat capacity of an electron gas

\[ \Delta U = \sum_0^\infty \epsilon d\epsilon \exp\left(\frac{-\beta\epsilon}{k_B T}\right) - \sum_0^\infty \epsilon \rho(\epsilon) \]

\[ \beta = \frac{1}{k_B T} \]

Density of states

Volume of the box

For particles in a box

\[ \rho(\epsilon) = \frac{V}{\left(\frac{2m}{\pi^2}\right)^{5/2}} \epsilon^{1/2} \]

We calculate the ground state energy

\[ U_0 = \frac{3}{5} N \epsilon F \]

Let us derive this result in a different way

\[ U(T) = \sum_0^\infty \epsilon d\epsilon \rho(\epsilon) f(\epsilon, T) \]

\[ = \sum_0^\infty \epsilon d\epsilon \frac{V}{\left(\frac{2m}{\pi^2}\right)^{5/2}} \epsilon^{1/2} \frac{1}{\beta(\epsilon - u) + 1} \]

\[ = \frac{\sqrt{2\pi}}{2\pi} \left(\frac{2m}{\pi^2}\right)^{5/2} \int_0^\infty \epsilon d\epsilon \frac{\epsilon^{1/2}}{\beta(\epsilon - u)} \frac{1}{\beta(\epsilon - u) + 1} \]

\[ = \frac{\sqrt{2\pi}}{2\pi} \left(\frac{2m}{\pi^2}\right)^{5/2} \int_0^\infty \epsilon d\epsilon \frac{\epsilon^{5/2}}{\beta(\epsilon - u)} \frac{1}{\beta(\epsilon - u) + 1} \]

Partial integration

\[ \text{Sharply peaked function for } T \to 0 \]

\[ \text{In the integrand we can approximate to leading order} \]

\[ \epsilon^{5/2} \approx \mu^{5/2} \]

Then

\[ U(T) = \frac{\sqrt{2\pi}}{2\pi} \left(\frac{2m}{\pi^2}\right)^{5/2} \frac{5}{5} \int_0^\infty \beta d\epsilon \frac{\epsilon^{5/2}}{\left(\epsilon^{5/2} + 1\right)^2} \]
Next, let us calculate the term of $o(C^2)$

$$U(c) = U(0) + \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\epsilon \frac{1}{\varphi^3} \frac{1}{\beta (\epsilon^m + 1)}$$

$$= U(0) + \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{1}{\beta^2} \int_0^\infty dx \frac{x^2}{(1 + x^2)^{3/2}}$$

$$= U(0) + \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\pi}{4}$$

We still need

$$U_0 = \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\pi}{4}$$

$$\Rightarrow U(c) = \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left[ U_0 + \frac{\pi}{4} \right]$$

We still need the relation between $U$ and $N$ to second order in $C$. This is what we will do next.

$$N = \int_0^\infty d\epsilon \varphi(\epsilon) \frac{1}{\beta (\epsilon - m) + 1}$$

$$= \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{\left[ \frac{\epsilon - m}{\beta} \right]^{1/2} + 1}$$

partial integration

$$N = \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{\left[ \frac{\epsilon - m}{\beta} \right]^{1/2} \left( 1 + \frac{\epsilon - m}{\beta} \right)^2}$$
\[ U(T) = \frac{V}{2\pi} \left( \frac{2m}{h^2} \right)^{3/2} \frac{e}{5} \int_{-\infty}^{\infty} dx \left( \frac{x}{e^x + 1} \right)^2 \]

\[ = \frac{\varepsilon F}{5} \text{ for } T \to 0. \]

\[ U(T) = \frac{V}{2\pi} \left( \frac{2m}{h^2} \right)^{3/2} \frac{e}{5} \varepsilon F \]

\[ = \frac{V}{2\pi} \left( \frac{2m}{h^2} \right)^{3/2} \frac{e}{5} \varepsilon F \left( \frac{\varepsilon F}{2\pi m} \right)^{3/2} \left( 3\pi^2 \frac{V}{N} \right) \]

\[ = \frac{3}{5} N \varepsilon F \]

To find the heat capacity, we have to expand \( U(T) \) in powers of \( T \). To this end, we expand

\[ \varepsilon = \mu + (\varepsilon - \mu) \frac{3}{2} \mu^2 + (\varepsilon - \mu) \frac{3}{2} \mu^2 \]

Let us first show that the contribution from the linear term vanishes.

\[ x = \beta (3 - \mu) \quad \beta \to 0 \quad \beta \to \infty \]

\[ = \beta (3 - \mu) \quad \frac{\beta \varepsilon}{(2 \beta (3 - \mu) + 1)^2} \]

\[ = \int_{-\infty}^{\infty} dx \left( \frac{2}{e^x + 1} \right)^2 \]

\[ = \frac{1}{\beta} \int_{-\infty}^{\infty} dx \frac{x}{(e^{\frac{x}{2\beta}} + e^{-\frac{x}{2\beta}})^2} = 0 \]
\[ S = \frac{1}{2} \int_{-1}^{1} \frac{d^2}{(x^2 + y^2 + z^2)^{3/2}} \]

For the zeroth and second term, we obtain

\[ N = \frac{1}{2} \int_{-1}^{1} \frac{d^2}{(x^2 + y^2 + z^2)^{3/2}} \]

We have a limit of integral is extended to -\infty.

Because of the integrand is sharply peaked,

\[ \int_{-\infty}^{\infty} \frac{d^2}{(x^2 + y^2 + z^2)^{3/2}} = 0 \]
\[
E_F = \mu \left( 1 + \frac{\pi^2}{12} \frac{\xi^2}{\mu^2} \right)^{\frac{2}{3}}
\]

\[
= \mu \left( 1 + \frac{\pi^2}{12} \frac{\xi^2}{\mu^2} \right)
\]

\[
= E_F \left( 1 - \frac{\pi^2}{12} \frac{\xi^2}{E_F} \right)
\]

To this order

\[
\Rightarrow U(2) = \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left[ \frac{2}{\beta} \left( 1 - \frac{\pi^2}{12} \frac{\xi^2}{E_F} \right) \frac{\xi^2}{E_F} + \frac{\xi^2}{E_F} \frac{1}{2} \right]
\]

we do not need to include the \( \xi^2 \) correction

\[
\Rightarrow U(2) = \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{3}{5} \xi E F E F \times
\]

\[
\left[ 1 - \frac{\pi^2}{12} \frac{\xi^2}{E_F} \frac{5}{2} + \frac{\xi^2}{E_F} \frac{\pi^2}{2} \frac{5}{2} \right]
\]

\[
= \frac{V}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{3}{5} \pi^2 \frac{N}{V} E_F \left[ 1 - \frac{\xi^2}{E_F} \left( \frac{5}{8} - \frac{5}{24} \right) \right]
\]

\[
= \frac{3}{5} N \xi E F \left[ 1 + \frac{\pi^2}{12} \frac{\xi^2}{E_F^2} \right]
\]

\[
\Rightarrow \xi V = \frac{3}{5} N \xi E F \frac{\pi^2}{12} \frac{\xi^2}{E_F^2} = \frac{\pi^2}{2} N \frac{\xi}{E_F}
\]
7c) Experimental situation

\[ E = \text{several eV} \]

electron density \( n_e = 10^{22} \text{ electrons/cm}^3 \)

Fermi velocity \( \frac{1}{2} m v_F^2 = E \)

\[ v_F = 10^8 \text{ m/s} \]

two contributions to specific heat:

- electrons \( \propto T \)
- lattice vibrations \( \propto T^3 \)

See Debye treatment

\[ \Rightarrow C_v = \gamma T + A T^3 \]

This describes the experimental data well

7f) White dwarf stars

It is believed that the interior of white dwarf stars is a degenerate electron gas.

Example: Sirius B, \( M = 2.10^{33} \text{ gr} \)

\[ R = 2 \times 10^8 \text{ cm} \]

\[ \rho = 0.7 \times 10^5 \text{ gr/m}^3 \]
\[ \varepsilon_F = 3 \times 10^5 \text{ eV} = 3 \times 10^9 \text{ MeV} \]

The interior temperature of white dwarfs is \( \leq 10^8 \text{ K} \)

\( \Rightarrow \) electron gas in highly degenerate

39) Nuclear matter

\[ R = 1.3 \times 10^{-13} \text{ m} \]

\[ n = \frac{A}{\frac{4}{3} \pi R^3} \approx 10^{38} \text{ cm}^{-3} \]

\( \Rightarrow \) \( n_F = n_n \approx \frac{1}{3} 10^{38} \text{ cm}^{-3} \)

\[ \varepsilon_F = \frac{h^2}{2M} \left( \frac{8 \pi^2 n}{3} \right)^{2/3} \leq 27 \text{ MeV} \]

\( \Rightarrow \) \( \frac{U_0}{N} = \frac{3}{5} \varepsilon_F = 16 \text{ MeV} \)

Kinetic energy per nucleon

This is a typical nuclear energy scale
Bose Einstein condensation

All particles of a Bose gas are in the ground state of the system.

\[
f(\epsilon, T) = \frac{1}{e^{(\epsilon - \mu)/kT} - 1}
\]

Occupancy of state

ground state at \( \epsilon = 0 \) \( \Rightarrow \) \( f(0, T) = \frac{1}{e^{(0 - \mu)/kT} - 1} \)

Occupancy should become macroscopic \( \Rightarrow \frac{\mu}{kT} \ll 1 \)

\[
\Rightarrow \frac{f(0, T)}{f(\epsilon, T)} = \frac{1}{1 - e^{-\epsilon/kT}} \approx \frac{\epsilon}{kT} \mathcal{N} \quad \text{size of particles}
\]

\[
\mu = -\frac{\epsilon}{N}
\]

Notice that the chemical potential has to be negative because otherwise the occupation of the ground state would become negative.

**Example:** Free gas \( \epsilon = \frac{kT}{2m} (n_x^2 + n_y^2 + n_z^2) \)

\[
\begin{align*}
H & = L = 1 \Rightarrow \frac{k^2}{2m} (\mathbf{\hat{p}})^2 \\
N & = 10^{22}, T = 1 \text{ mK}
\end{align*}
\]

\( \Rightarrow \) For condensation \( \mu = -1.4 \times 10^{-3} \text{ a.u.} \)

\[
\frac{f_1}{f} = \frac{e^{\epsilon - \mu}}{e^{\epsilon/kT} - 1} = \frac{1}{\mathcal{E}_1} \approx \frac{1.4 \times 10^{-3}}{2.4 \times 10^{-3}} = 0.6
\]

\[
\approx 6 \times 10^{10}
\]

So that only an extremely small fraction of the particles is in the first excited state.
Density of states: \( p(\varepsilon) = \frac{\sqrt{2m\varepsilon}}{\pi \hbar^2} \)

\[
N = F_0 + \sum_{n=0}^\infty F_n
\]

\[
= N_0(\varepsilon) + \int_{\varepsilon_0}^{\infty} p(\varepsilon) f(\varepsilon) d\varepsilon
\]

No of particles in the condensed phase:

\[
N_0(\varepsilon) = \frac{1}{\lambda - 1}
\]

\[
N_0(\varepsilon) = \frac{\sqrt{2m\varepsilon}}{4\pi\hbar^2} \int_{\varepsilon_0}^{\infty} \frac{x^{1/2} dx}{x^{1/2} - 1}
\]

\[
\frac{\varepsilon}{\epsilon} = \lambda
\]

\[
= \frac{\sqrt{2m\varepsilon}}{4\pi\hbar^2} \frac{3\lambda}{2} \int_{\varepsilon_0}^{\infty} \frac{x^{1/2} dx}{x^{1/2} - 1}
\]

For \( N_0 \) to be of order \( N \), \( \lambda \) should be very close to 1. That means, in the excited state integral we can put \( \lambda = 1 \)

\[
N_0(\varepsilon) = \frac{\sqrt{2m\varepsilon}}{4\pi\hbar^2} \frac{3\lambda}{2} \int_{\varepsilon_0}^{\infty} \frac{x^{1/2} dx}{x^{1/2} - 1}
\]

\[
= \frac{\sqrt{2m\varepsilon}}{4\pi\hbar^2} \frac{3\lambda}{2} \int_{\varepsilon_0}^{\infty} \frac{x^{1/2} dx}{x^{1/2} - 1}
\]

\[
= \frac{1.306}{\sqrt{\pi}} \left( \frac{2m\varepsilon}{\pi \hbar^2} \right)^{3/2} = 2.612 n a \quad \varepsilon_0 = \left( \frac{1.21 \pi}{2\pi} \right)
\]
condensation temperature

temperature above which all atoms are
in the normal phase, i.e.

\[ N_e (T_e) = N \Rightarrow \frac{1.306}{4} \left( \frac{2.0}{\pi^2} \right)^{3/2} T_e^{3/2} = 1 \]

\[ T_e = \frac{2 \pi^2 k}{M} \left( \frac{N}{2.012 V} \right)^{3/2} \]

\[ \frac{N_e}{N} = \left( \frac{T_c}{T_e} \right)^{3/2} \]

\[ \frac{N_e}{N} = 1 - \left( \frac{T_c}{T_e} \right)^{3/2} \]

Liquid Helium: \( T_e = \frac{115}{V_m^{3/2}} R \)

\( V_m \) in volume of 1 mole \( V_m = 23.6 \text{ cm}^3 \)

\( \rho \) in molecular weight \( \rho = 4 \)

\( T_e = 3.14 \) K

experimental value 2.17 K
\( ^4\text{He} \) and superfluidity.

- Liquid \(^4\text{He}\) behaves more like a gas than a liquid.
  \[ V_p = 27.6 \text{ cm}^3 \] whereas for a static lattice it would be \( V_p = 9 \text{ cm}^3 \).
- Zero point fluctuations are important.
- Liquid only exists below 5.2 K.

Phase diagram:

- The phase diagram of \(^4\text{He}\) is completely different.
- Low-energy excitation of \(^4\text{He}\).

They are not free particle excitations with
\[ E_k = \frac{k^2}{2m} \] but longitudinal sound waves with
\[ E_k = \frac{1}{2} k v \] speed of sound.
2-29-05

Critical temperature $T_c = \frac{115}{\sqrt{R}} \left( \frac{2}{3} \right) \frac{1}{22.6 \text{ cu}^3/\text{mol}}$

7e) Low-energy excitations of $^5\text{He}$
8e) Heat and work
9e) Cannot inequality
These excitations are also called quasi-particles.

Let us consider the motion of a particle in a super-fluid and see how it can loose energy to the fluid by exciting the fluid to end with momentum \( \mathbf{u} \).

\[
M_0 \mathbf{u} \rightarrow \mathbf{u} + \mathbf{t} \mathbf{u} + \mathbf{E}_u \\
\Rightarrow \frac{1}{2} M_0 \mathbf{u}^2 = \frac{1}{2} M_0 \mathbf{u}^2 + \mathbf{E}_u \\
M_0 \mathbf{u} = \mathbf{u}' + \mathbf{t} \mathbf{u}
\]

\[
\Rightarrow (M_0 \mathbf{u})^2 = (M_0 \mathbf{u} - \mathbf{t} \mathbf{u})^2 \\
\Rightarrow M_0^2 \mathbf{u}^2 = M_0^2 \mathbf{u}^2 + t^2 \mathbf{u}^2 - 2 M_0 t \mathbf{u} \cdot \mathbf{u} \\
\Rightarrow \mathbf{E}_u M_0 = t^2 \mathbf{u}^2 - 2 M_0 \mathbf{u} \cdot \mathbf{u} \\
\Rightarrow t^2 \mathbf{u}^2 - \frac{1}{2} M_0 t^2 \mathbf{u}^2 = \mathbf{E}_u \\
\Rightarrow \exists \text{ minimum velocity to dissipate energy}
\]

\[
\frac{v_{\text{min}}}{M_0} = \frac{\mathbf{E}_u + \frac{1}{2} t^2 \mathbf{u}^2}{t \mathbf{u}}
\]

Large mass limit

\[
\left| v_{\text{min}} \right| = \frac{\mathbf{E}_u}{t \mathbf{u}}
\]

No dissipation occurs below this velocity.

This means that the viscosity is zero.
For phonons \( \varepsilon_k = U_s + K \)

\[ \Rightarrow U_{\text{min}} = U_s \]

For free particle excitation \( \varepsilon_k = \frac{k^2 v^2}{2M} \)

\[ \Rightarrow U_{\text{min}} = \frac{k^2 v^2}{2M} \rightarrow 0 \quad \text{for } k = 0 \]

Experimentally it turns out that \( U_{\text{min}} \) is much less than the velocity of sound.

\[ \varepsilon_k = k v U_s \]

\[ \varepsilon_k = 0 + k \]

\[ \Rightarrow U_{\text{min}} = \frac{k v U_s}{k_0} \]

Dissipation can occur if the dashed line and the curve intersect.

\[ \Rightarrow U_{\text{min}} = \frac{\dot{Q}}{k_0} \approx 5 \times 10^6 \text{ cm/sec} \]