GRASSMANN INTEGRATION IN STOCHASTIC QUANTUM PHYSICS: THE CASE OF COMPOUND–NUCLEUS SCATTERING

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Abstract:

Using a stochastic model for \( N \) compound-nucleus resonances coupled to the channels, we calculate in the limit \( N \to \infty \) the ensemble average of the S-matrix (the "one-point function"), and of the product of an S-matrix element with the complex conjugate of another, both taken at different energies (the "two-point function"). Using a generating function involving both commuting and anticommuting integration variables, we evaluate the ensemble averages trivially. The problem of carrying out the remaining integrations is solved with the help of the Hubbard–Stratonovitch transformation. We put special emphasis on the convergence properties of this transformation, and on the underlying symmetries of the stochastic model for the compound nucleus. These two features together completely define the parametrization of the composite variables in terms of a group of transformations. This group is compact in the "Fermion–Fermion block" and non-compact in the "Boson–Boson block". The limit \( N \to \infty \) is taken with the help of the saddle-point approximation. After integration over the "massive modes", we show that the two-point function can be expressed in terms of the transmission coefficients. In this way we prove that the fluctuation properties of the nuclear S-matrix are the same over the entire spectrum of the random Hamiltonian describing the compound nucleus. The integration over the saddle-point manifold is carried out using symmetry properties of the random Hamiltonian. We finally obtain a closed-form expression for the two-point function in terms of a threefold integral over real variables. This expression can be easily evaluated numerically.

1. Introduction and overview

Random Hamiltonians are widely used in theoretical physics for the modelling of statistical phenomena. The implementation of such concepts often poses serious problems, however, because observables usually depend in a highly non-linear fashion on the random Hamiltonian, this making the calculation of ensemble averages very difficult. In the present paper, we present the exact (non-perturbative) solution to a long-standing problem in the statistical theory of nuclear reactions, which dates back to Niels Bohr’s model of the compound nucleus. This problem, too, has been formulated in terms of a random Hamiltonian. The solution is here obtained by combining the use of a generating function with the method of integration over anticommuting or Grassmann variables. There are other problems in statistical nuclear theory which have previously been solved exactly. We recall the celebrated results of Dyson and Mehta on spectral fluctuation properties [1]. The method of solution presented in this paper differs from earlier approaches in that we use methods which are widely applied in statistical physics and elsewhere. Our solution will therefore be of interest beyond the confines of nuclear physics, and it may serve as a case study in stochastic quantum physics. We believe that two issues in particular deserve general interest. One issue relates to the parametrization of the composite variables. After taking the ensemble average of the two-point function, we introduce these variables via the Hubbard–Stratonovitch transformation in the usual fashion. In the context of the replica trick, where one proceeds similarly, the problem of how to parametrize the composite variables also arises. For ordinary (commuting) integration variables, this problem has been thoroughly discussed, and solved, by Schäfer and Wegner [2]. These authors demonstrated the necessity to parametrize the composite variables in terms of a non-compact group. We are not aware of an investigation of the same problem in the presence of both commuting and anticommuting integration variables, and we believe that our results relating to this point [3] are novel. The commuting composite variables form two classes. For one class (the "Boson–Boson block"), the arguments of Schäfer and Wegner prevail, and the parametrization must make use of a non-compact group. For the other class (the "Fermion–Fermion block") we show that it is mandatory in contradistinction to parametrize the variables in terms of a compact group. We are led to this conclusion using arguments of convergence and symmetry which are perfectly general and independent of the specific nuclear-physics problem which we solve. The second issue relates to the manner in which the exact solution is derived. It turns out that the nuclear-physics problem here investigated is comparatively simple: After introduction of the composite variables and integration over the "massive modes", the problem is mapped onto a non-linear \( \sigma \)-model of dimensionality zero. Nevertheless, the evaluation of the remaining 16 integrals is not trivial. It becomes possible only by
tenaciously using the symmetry properties of the model. It is for this reason that we believe that also this second issue, and the methods used to calculate the answer, are of general interest and hopefully useful in another context. In writing the paper, we have therefore aimed at a compact but full account of all steps taken.

The problem which we consider concerns the calculation of average compound-nucleus cross sections. After early work by N. Bohr, Bethe and Weisskopf, and motivated by Wigner's idea of describing properties of highly excited nuclear levels in terms of a random Hamiltonian, various authors in the 1950's and 60's defined this problem in terms of an ensemble of random matrices coupled to a (fixed) number of channels [4]. The random matrices of dimension $N$ (we here eventually take the limit $N \to \infty$) simulate the presence of $N$ compound-nucleus resonances. Following Wigner, one usually chooses the ensemble as the Gaussian Orthogonal Ensemble (GOE). In spite of numerous efforts, this model could not be solved exactly, and only partial answers in terms of series expansions (valid when the number of open channels is either very small, or very large) have been obtained [4]. We only mention in passing that this model is of interest not only as a problem of theoretical physics but also in nuclear-science applications.

Using the GOE to simulate the stochastic properties of highly excited nuclear levels may seem pure phenomenology, and even arbitrary. Recent developments in the theory of non-integrable classical systems and their quantum counterparts suggest that this is not the case, and have greatly enhanced our enthusiasm for this work. Numerical studies of particularly simple classical chaotic systems, and of the distribution and fluctuation properties of the eigenvalues of their quantum counterparts, have shown [5] that these fluctuation properties coincide (within statistical accuracy) with those found experimentally in high-lying parts of nuclear spectra, and with those predicted by the GOE. This observation suggests that nuclear fluctuation properties are a universal feature of microscopic quantum systems: They should surface for any system in which a resolution of properties down to an energy scale given by the average level spacing is experimentally feasible, and they should hence be generic, expressing a property which in the analogous classical system would be identified as chaotic behaviour. The use of the GOE to simulate these fluctuation properties is not arbitrary. The GOE can be derived [6] from the assumption of minimal information content of the Hamiltonian (an assumption which intuitively relates to classical chaotic motion); fluctuation properties obtained from the GOE are statistically indistinguishable from those generated by related matrix ensembles [1]; and the GOE is much more suitable for analytical treatment than are these other ensembles. For these reasons, we believe that the GOE encapsulates a very essential aspect of the nuclear dynamics, which is complementary to the description of regular features and of average nuclear properties as furnished by the shell model, the collective model, and similar approaches based on a mean-field description. We use the GOE with confidence for the description of fluctuation properties although it is known to fail badly for the description of average properties like the average level density. We also believe that our results are generally valid and insensitive to specific details of the GOE. This belief is supported by a recent and yet unpublished comparison (suggesting complete agreement) which we have carried out between numerical results obtained in the framework of our approach, and those obtained by the Mexican group [7]. These latter authors derived the probability distribution function of the nuclear S-matrix elements directly from a maximum entropy principle for the S-matrix, avoiding the introduction of a Hamiltonian altogether. In this sense we feel that the problem studied in the present paper lies at the interface between the classical mechanics of non-integrable systems and their quantum counterparts (these systems furnish the justification for our model), the statistical mechanics of disordered systems (these systems furnish much of the theoretical framework which we use although the disorder studied there is usually of geometrical rather than inherently dynamical origin), and fluctuation properties of microscopic quantum systems.
It goes without saying that our method can also be used to calculate other statistical properties of nuclei than the compound-nucleus cross section studied here. We hope in fact that it will be used widely. The method clearly provides a unified description of the fluctuation properties of nuclear spectra, and of the stochastic aspects of nuclear reactions. It is more powerful than the replica trick which we used previously [8, 9, 10], the latter only being able to yield asymptotic expansions [3]. In this paper, we focus attention mainly on the two-point function which we evaluate in the limit $N \to \infty$. The same method (when applied to the one-point function) yields simple integral representations for every finite $N$. This may be of interest in some cases. The calculation of $n$-point functions for $n > 2$ in the framework of the present approach poses problems which seem to grow quickly with increasing $n$.

The main body of the paper involves fairly technical manipulations of sets of both commuting and anticommuting integration variables. Except for parts of appendix L, we have aimed at a presentation which is self-contained. We felt that this would be desirable especially for nuclear theorists who are largely unfamiliar with the techniques used here, but also for a wider group of readers since we are not aware of any sufficiently detailed work on the use of anticommuting integration variables. In order to cope with a fairly sizeable amount of mathematical detail, in order to make the main body of the paper easy reading for the experts, and in order to clearly display the main line of reasoning, we have relegated all technical details to a number of appendices.

The model for the nuclear $S$-matrix is defined in section 2. The generating function defined in terms of commuting and anticommuting variables is introduced in section 3; cf. also appendices A and B. The ensemble average is calculated in section 4 and appendix C. In the same section, we also introduce the Hubbard-Stratonovitch transformation. Questions of convergence of this transformation, of the symmetry of a fundamental quadratic form, and of the proper parametrization of the composite variables form the subject of the central section 5 and appendices D and E. The integration over "massive modes" (possible in the limit $N \to \infty$) is carried out in section 6 and appendix F. In appendix G we show that the unitarity of the $S$-matrix is preserved by the steps taken up to and including section 6. In section 7 and appendix H, we show that the average two-point function can be expressed in terms of the transmission coefficients. In section 8 and appendices I, K and L, we carry out the remaining 8 integrations over anticommuting variables, and 5 of the remaining 8 integrations over commuting variables, finally arriving at an integral representation for the two-point function in terms of a threefold integral which is easily evaluated numerically. Section 9 contains a summary, and our conclusions.

In dealing with anticommuting variables, we received much stimulation from a recent paper by Efetov [11]. Although pretending to be a review, this paper unfortunately contains so little detail that we found it of no help for our actual calculations. This situation furnished us with yet another motivation to write our paper in its present form.

2. The model

For fixed spin and parity, we consider a nuclear scattering problem involving compound-nucleus formation. The physical channels (denoted by small latin letters $a, b, c, \ldots$) are characterized, as usual, by the internal states of the two fragments, the channel spin, and the relative angular momentum. We denote by $E$ the total energy of the system. The channel wave functions are called $|\chi_a(E)\rangle$, and we use the normalization that $\langle \chi_a(E_1) | \chi_b(E_2) \rangle = \delta_{ab} \delta(E_1 - E_2)$. The compound-nucleus resonances are caused by a large number $N \gg 1$ of bound states $|\mu\rangle$, $\mu = 1, \ldots, N$ with $\langle \mu | \nu \rangle = \delta_{\mu \nu}$. We shall later take the limit $N \to \infty$. In the space spanned by the functions $\{|\mu\rangle, |\chi_a(E)\rangle\}$, the model Hamiltonian $\mathcal{H}$ has the form
\[ \mathcal{H} = \sum_{a} \int_{\epsilon_a}^{\infty} dE \, \chi_a(E) \langle \chi_a(E) \rangle + \sum_{\mu, \nu} |\mu\rangle \langle \nu| H_{\mu\nu} \langle \nu| + \sum_{\mu, \nu} \sum_{a} \left\{ |\mu\rangle \int_{\epsilon_a}^{\infty} dE \, W_{\mu a}(E) \langle \chi_a(E) \rangle + h.c. \right\} \]. (2.1)

Here, \( \epsilon_a \) is the threshold energy in channel \( a \). The first term on the r.h.s. of eq. (2.1) is the channel part of the Hamiltonian. We have simplified the model by dropping all terms that would couple different channels with each other. (The inclusion of such direct reactions would complicate the formalism, see the comments below.) To simplify the problem further, we put all the elastic phase-shifts equal to zero. (This simplification is permissible because in any case, all our final expressions would contain the phase-shifts only in a trivial fashion.) The second term in eq. (2.1) describes the interaction of the bound states with each other. Time-reversal invariance and Hermitecity allow us to choose the \( N \times N \) matrix \( H_{\mu\nu} \) real and symmetric. The third term describes the coupling between levels and channels; again, we can choose the matrix elements \( W_{\mu a}(E) \) real.

The scattering matrix \( S_{ab}(E) \) of the Hamiltonian \( \mathcal{H} \) can be worked out using standard techniques (cf., for instance, eq. (4.2.24) of ref. [12]) and is given by

\[ S_{ab}(E) = \delta_{ab} - 2i\pi \sum_{\mu, \nu} W_{\mu a}(E) (D^{-1})_{\mu\nu} W_{\nu b}(E) \] (2.2)

where

\[ D_{\mu\nu} = E \delta_{\mu\nu} - H_{\mu\nu} - F_{\mu\nu}(E) \] (2.3)

and, with \( E^+ = E + i\eta \) and \( \eta \) positive infinitesimal,

\[ F_{\mu\nu}(E) = \sum_{a} \int_{\epsilon_a}^{\infty} dE' \, \frac{W_{\mu a}(E') \, W_{\nu a}(E')}{E^+ - E'}. \] (2.4)

We are interested in a situation where the mean spacing between compound-nucleus resonances is very small in comparison with the mean spacing between channel thresholds. We therefore neglect all threshold effects including the dependence of the \( W_{\mu a} \) on energy, and replace the \( F_{\mu\nu}(E) \) by their imaginary parts,

\[ F_{\mu\nu} \equiv -i\pi \sum_{a \text{ open}} W_{\mu a} W_{\nu a}, \] (2.5)

dropping the principal-value integral. The sum in eq. (2.5) extends over the \( A \) open channels only, and so do all sums over channels occurring in this and subsequent sections. The simplification (2.5) is introduced for convenience only. We show in appendix H that inclusion of the energy-dependence of the \( W_{\mu a} \) and inclusion of the shift functions, is straightforward and does not change the form of the final result.

The \( S \)-matrix (2.2) is a symmetric matrix of dimension \( A \). A stochastic element must be introduced into \( S_{ab} \) if we wish to describe the random fluctuations associated with compound-nucleus scattering.
We postulate that the quasibound levels, obtained by diagonalising $H_{\mu\nu}$, have the same statistical properties as are experimentally observed for isolated resonances [1]. We accordingly take the matrix $H_{\mu\nu}$ to be a member of an ensemble of matrices, the Gaussian Orthogonal Ensemble (GOE). The GOE is characterized by the statement that the independent elements of the matrix $H_{\mu\nu}$ (i.e. those with $\mu \geq \nu$) are uncorrelated random variables with a Gaussian probability distribution centered at zero and second moments given by

$$H_{\mu\nu}H_{\mu'\nu'} = \frac{\lambda^2}{N} (\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\mu'\nu}).$$  \hspace{1cm} (2.6)

Here, the bar denotes the ensemble average, and $\lambda$ is a strength parameter. For $W_{\mu a} \equiv 0$ (all $\mu$ and $a$), the GOE yields in the limit $N \to \infty$ a spectrum with a level density given by the semicircle law [1]. The factor $N^{-1}$ on the r.h.s. in eq. (2.6) is inserted in order to keep the radius of the semicircle finite in the limit $N \to \infty$.

The ensemble of matrices (2.6) defines via eq. (2.2) an ensemble of $S$-matrices. In this paper, we evaluate the ensemble averages $S_{ab}(E)$ (the average “one-point function”), and $S_{ab}(E_1)S_{cd}(E_2)$ (the average “two-point function”). We shall succeed in expressing the average two-point function in terms of the average one-point function (or, equivalently, the fluctuations in terms of the mean values). In compound–nucleus theory, the average $S$-matrix elements (which relate to fast processes involving few degrees of freedom) are usually considered as input and are given in terms of an optical model or a coupled-channels model. The task consists in calculating average cross sections from this input. This is what we proceed to do.

We calculate ensemble averages at fixed energy. The observables are energy averages taken for a fixed member of the ensemble. This raises the problem of ergodicity, i.e., equality of the two averages. Ergodicity can be ascertained using the four-point function [1]. Knowledge of the four-point function is also required to calculate fluctuations of cross sections about their average values. Because of the considerable complexity encountered in evaluating even the average two-point function, we have so far not calculated the average four-point function. We have no doubt, however, that the two-point function is ergodic.

It may appear that with the coupling matrix elements $W_{\mu a}$, a large number ($A \cdot N$) of parameters has been introduced. It turns out, however, that both the average one-point function and the average two-point function depend only upon the quantities $N^{-1}\sum_{\mu} W_{\mu a} W_{\mu b}$. This is a consequence of the orthogonal invariance of the GOE. There are as many such quantities as there are elements of $S_{ab}$. This is why it is possible to express the average two-point function in terms of the average one-point function without additional parameters.

We have remarked above that eq. (2.1) implies the omission of all direct couplings between the channels. This is also manifest from the form (2.2) of the scattering matrix which becomes equal to the unit matrix in the absence of compound–nucleus scattering. It is consistent with the absence of direct reactions to require that the average $S$-matrix be diagonal, $S_{ab} = \delta_{ab} S_{aa}$. Anticipating later developments (cf. especially eq. (7.7)), we remark that this condition is equivalent to the requirement that

$$N^{-1}\sum_{\mu} W_{\mu a} W_{\mu b} = \delta_{ab} v_a^2.$$  \hspace{1cm} (2.7)

This equation defines $v_a^2$. To simplify the calculations, we use the condition (2.7) from the outset.
The omission of direct reactions is not as stringent an assumption as it may seem. Indeed, it is known [13] that for a non-diagonal average $S$-matrix $S_{ab}$, a unitary transformation $U$ can be found such that $(USU^T)_{ab}$ (with $U^T$ the transpose of $U$) is diagonal on average, and has the same fluctuation properties as the $S$-matrix of a problem without direct reactions. Except for this unitary transformation, the problem defined by eqs. (2.2), (2.3), (2.5), (2.6) and (2.7) thus encompasses the general fluctuation problem of compound-nucleus theory.

It is useful to connect the problem formulated above with the standard approach to compound-nucleus theory. For this purpose we introduce the orthogonal matrix $O_{\mu\nu}$ which diagonalises $H_{\mu\nu}$

$$(OHO^T)_{\mu\nu} = e_\mu \delta_{\mu\nu},$$

where the superscript $T$ denotes the transpose, and we write

$$v_\mu^a = \sum_\nu O_{\mu\nu} W_{\nu a}.$$  

(2.8)

(2.9)

In the formulation (2.2), the matrix elements $W_{\mu a}$ are fixed (non-stochastic) quantities; the entire stochasticity of the problem resides in the GOE matrix $H_{\mu\nu}$. After the transformation (2.8), (2.9) both the eigenvalues $e_\mu$ and the matrix elements $v_\mu^a$ are random variables. The analysis of the GOE shows [1] that the $e_\mu$ are uncorrelated with the $v_\mu^a$ for all values of $\mu$, $\nu$ and $a$, that the $v_\mu^a$ are Gaussian-distributed with mean value zero, and that the $e_\mu$ have a complicated probability distribution characterized by Wigner repulsion and stiffness of the spectrum. Using the transformation (2.8), (2.9), we can cast the $S$-matrix (2.2) into the form

$$S_{ab} = \left( \frac{1 - iK}{1 + iK} \right)_{ab}$$

(2.10)

where the ensemble of $K$-matrices is given by

$$K_{ab} = \pi \sum_\mu \frac{v_\mu^a v_\mu^b}{E - e_\mu}.$$ 

(2.11)

Equations (2.10) and (2.11) are the starting point of most of the work [4] on the problem done previously. (The $R$-matrix [14] approach leads to an equivalent formulation.) This starting point, although equivalent to eq. (2.2), suffers from the drawback that the introduction of the eigenvalues $e_\mu$ is possible only at the expense of giving up the manifestly orthogonally invariant formulation of the theory. We ascribe the difficulties encountered previously in evaluating the average two-point function partly to this circumstance. We shall see, in fact, that the present approach keeps and utilizes the orthogonal invariance of the GOE (or its implications) all the way through, virtually to the last step of the calculation. We believe that this is its main strength.

3. Representation of the one- and two-point functions in terms of Gaussian integrals involving Grassmann variables

In a previous paper using the replica trick [10], the $S$-matrix was written as the logarithmic derivative of a generating function. The replica trick can be avoided, and closed expressions for the average one-
and two-point functions can be obtained, if we succeed in normalizing the generating function to unity. This is possible with the help of anticommuting variables. In this section, we introduce the basic formulas and a compact notation.

Equation (2.2) shows that the random variables are contained entirely in the propagator \((D^{-1})_{\mu\nu}\). It therefore suffices to construct a generating function for this matrix. This is done using the results of appendix B.

We introduce two independent symmetric source matrices \(J^C_{\mu\nu}\) and \(J^A_{\mu\nu}\), and we define

\[
D_{\mu\nu}(J^C) = D_{\mu\nu} - J^C_{\mu\nu}; \quad D_{\mu\nu}(J^A) = D_{\mu\nu} + J^A_{\mu\nu}.
\tag{3.1}
\]

Here \(D_{\mu\nu}\) is given by eqs. (2.3) and (2.5). The energy \(E\) is given a small positive imaginary part \(\eta > 0\) to insure convergence of the integrals defined below. We define a \(4N \times 4N\) graded matrix \(D'\) by what is henceforth called “block construction”: We put the four \(N \times N\) matrices \(D(J^C), D(J^A), D(J^\lambda), D(J^\lambda')\) defined by eqs. (3.1) in this order into the diagonal blocks of \(D'\), and write zeros everywhere else. As in appendix A, we introduce \(2N\) ordinary real integration variables \(S_1, \ldots, S_N, S'_1, \ldots, S'_N\) and \(2N\) anticommuting integration variables \(\chi_1, \ldots, \chi_N, \chi'_1, \ldots, \chi'_N\) which together form the elements of the graded vector \(\phi\). Using the definitions (A.8), (B.7) and (B.10), we consider the generating function

\[
Z(E, J) = \int d[\phi] \exp\{L(\phi, J)\},
\tag{3.2}
\]

where the “Lagrangian” is given by

\[
L = \frac{i}{2}(\phi^t D' \phi).
\tag{3.3}
\]

The expression \((\phi^t D' \phi)\) is a short-hand notation for the scalar product of the vector \(D' \phi\) and the vector \(\phi^*\). Equations (B.3) and (B.5) allow us to evaluate \(Z(E, J)\) explicitly; we find

\[
Z(E, J) = \det(D(J^\lambda)) [\det(D(J^C))]^{-1}.
\tag{3.4}
\]

This shows that for \(J^C = 0 = J^\lambda\) the function \(Z(E, J)\) is normalized to unity,

\[
Z(E, 0) = 1.
\tag{3.5}
\]

Normalization to unity as in eq. (3.5) makes it possible to calculate the average one- and two-point functions analytically.

Using the identity \(\det(A) = \exp\{\text{tr} \ln A\}\), we find

\[
\frac{\partial}{\partial J^C_{\mu\nu}} \ln[\det(D(J^C))]^{-1} \bigg|_{J^C=0} = \frac{\partial}{\partial J^\lambda_{\mu\nu}} \ln \det(D(J^\lambda)) \bigg|_{J^\lambda=0} = (2 - \delta_{\mu\nu})(D^{-1})_{\mu\nu}.
\tag{3.6}
\]

Combining this result with eq. (3.4), we have

\[
\frac{\partial}{\partial J^C_{\mu\nu}} Z(E, J) \bigg|_{J^C=0} = \frac{\partial}{\partial J^\lambda_{\mu\nu}} Z(E, J) \bigg|_{J^\lambda=0} = (2 - \delta_{\mu\nu})(D^{-1})_{\mu\nu}.
\tag{3.7}
\]
Another form for \((D^{-1})_{\mu \nu}\) is found if we consider the matrices \(J^A\) and \(J^C\) as identical and write \(J^{\mu \nu} = J^A_{\mu \nu} = J^C_{\mu \nu}\). Then,

\[
(1 - \frac{1}{2}\delta_{\mu \nu}) (D^{-1})_{\mu \nu} = \frac{1}{4} \frac{\partial}{\partial J_{\mu \nu}} Z(E, J) \bigg|_{J=0}.
\]

(D.8)

Differentiating eq. (3.2), we find the equivalent formula

\[
(D^{-1})_{\mu \nu} = \frac{i}{4} \int d[\phi] \exp\{\mathcal{L}(\phi, 0)\} \left[ \chi^*_\mu \chi_\nu + \chi^*_\nu \chi_\mu - S^1_\mu S^1_\nu - S^2_\mu S^2_\nu \right].
\]

(D.9)

This expression is manifestly symmetric in \(\mu\) and \(\nu\).

Equations (3.7)–(3.9) constitute the desired result. In the calculation of the two-point function, we must evaluate products of the elements of two inverse \(D\)-matrices. The large number of variables and symbols then encountered calls for a notational simplification. This forms the last part of the present section.

Let \(E(1)\) and \(E(2)\) be the energy arguments of the two \(S\)-matrix elements, and let the argument \(p\) with \(p = 1, 2\) distinguish terms relating to \(Z(E(1))\) and to \(Z^*(E(2))\), respectively. This applies, in particular, to the source matrices \(J(p)\), to the ordinary integration variables \(S_\mu(p), i = 1, 2; \mu = 1, \ldots, N\), and to the anticommuting integration variables \(\chi_\mu(p), \mu = 1, \ldots, N\), and their complex conjugates. Both \(E(1)\) and \(E(2)\) are given a positive infinitesimal part \(\eta\) which is chosen independent of \(p\) for simplicity. We proceed to introduce a generating function which equals the product \(Z(E(1), J(1))Z^*(E(2), J(2))\) but has the compact form of eq. (3.2).

To this end, we define a graded vector \(\psi\) with \(8N\) elements by writing down \(\psi^T\),

\[
\psi^T = ([S^1_\mu(1)], [S^2_\mu(1)], [S^1_\mu(2)], [S^2_\mu(2)], [\chi_\mu(1)], [\chi^*_\mu(1)], [\chi_\mu(2)], [\chi^*_\mu(2)]).
\]

(3.10)

The associated volume element is

\[
d[\psi] = d[S(1)] d[S(2)] d[\chi(1)] d[\chi(2)].
\]

(3.11)

To account for the complex conjugation sign in the terms with \(p = 2\), we introduce the \(8 \times 8\) diagonal matrix \(L_{\alpha \beta}\) with diagonal elements \((1, 1, -1, 1, 1, 1, -1, 1)\) in this sequence; the \(8N \times 8N\) graded matrix \(L\) as the direct product \(L_{\alpha \beta} \delta_{\mu \nu}\) with \(1 \leq \alpha, \beta \leq 8\) and \(1 \leq \mu, \nu \leq N\); and we define with \(1 \leq \alpha, \beta \leq 8\) and \(1 \leq \mu, \nu \leq N\)

\[
E = \frac{1}{2}(E(1) + E(2)); \quad \epsilon = E(2) - E(1)
\]

(3.12a)

and

\[
E = E\{\delta_{\alpha \beta} \delta_{\mu \nu}\}; \quad \mathcal{G} = \epsilon \cdot L;
\]

\[
H = \{\delta_{\alpha \beta} H_{\mu \nu}\}; \quad W = \pi \cdot \left\{ L_{\alpha \beta} \sum_a W_{\mu a} W_{\nu a} \right\};
\]

\[
\delta = \eta \cdot L.
\]

(3.12b)
The graded matrix $J$ is defined by block construction, with

$$\{-J_{\mu
u}(1), -J_{\mu
u}(2), J_{\mu
u}(1), J_{\mu
u}(2)\}$$

(3.12c)

as diagonal blocks. It is useful to introduce a $8 \times 8$ diagonal graded matrix $I$ which carries the same signs as the expression (3.12c), i.e., $I$ has the diagonal elements $(-1, -1, -1, 1, 1, 1, 1, 1)$.

Using these definitions, we write

$$D(J) = (E - H + i\delta + iW + J - \frac{1}{2} \mathcal{B})$$

(3.13)

and have

$$Z(E(1), E(2), J) = \int d[\psi] \exp\{\mathcal{L}(\psi, J)\}$$

(3.14)

with

$$\mathcal{L}(\psi, J) = \frac{i}{2} (\psi' L^{1/2} D(J) L^{-1/2} \psi) .$$

(3.15)

The matrices $(D^{-1})_{\mu\nu}$ etc. are now given by

$$\left(1 - \frac{1}{2} \delta_{\mu\nu}\right) [D^{-1}(E(1))]_{\mu\nu} = \frac{1}{4} \frac{\partial}{\partial J_{\mu\nu}(1)} Z(E(1), E(2), J) \bigg|_{J=0} ,$$

(3.16a)

$$\left(1 - \frac{1}{2} \delta_{\mu\nu}\right) [D^{*\nu^{-1}}(E(2))]_{\mu\nu} = \frac{1}{4} \frac{\partial}{\partial J_{\mu\nu}(2)} Z(E(1), E(2), J) \bigg|_{J=0} ,$$

(3.16b)

$$[1 - \frac{1}{2} \delta_{\mu(1)\nu(1)}] \left[1 - \frac{1}{2} \delta_{\mu(2)\nu(2)}\right] [D^{-1}(E(1))]_{\mu(1)\nu(1)} [D^{*\nu^{-1}}(E(2))]_{\mu(2)\nu(2)}$$

$$= \frac{1}{16} \frac{\partial^2}{\partial J_{\mu(1)\nu(1)}(1) \partial J_{\mu(2)\nu(2)}(2)} Z(E(1), E(2), J) \bigg|_{J=0} .$$

(3.16c)

We note that $Z(E(1), E(2), J)$ contains the entire information relevant for our problem. Using the steps which lead to eq. (3.9) we also have

$$[D^{-1}(E(1))]_{\mu(1)\nu(1)} [D^{*\nu^{-1}}(E(2))]_{\mu(2)\nu(2)} = \frac{1}{16} \int d[\psi] \exp\{\mathcal{L}(\psi, 0)\}$$

$$\times \prod_{p=1}^{2} [\chi_{\mu(p)}(p) \chi_{\nu(p)}(p) + \chi_{\nu(p)}(p) \chi_{\mu(p)}(p)$$

$$- S_{\mu(p)}(p) S_{\nu(p)}(p) - S_{\mu(p)}(p) S_{\nu(p)}(p)] .$$

(3.17)
Although the results (3.16) and (3.17) with $Z$ given by eq. (3.14) have been derived for the scattering problem, the general form of $Z$ and of the relation between $Z$ and the observables is quite universally valid. It is only the terms $W$ and $J$ in the matrix $D(J)$ which change if other GOE problems are considered. For example, the matrix element $[(E(1) - H)^{-1}]_{\mu\nu}$ is obtained by putting $W = 0$ in eq. (3.13); the expressions $\mathrm{tr}[E(1)^{-1}]$ and $\mathrm{tr}[E(1)^{-1}] \mathrm{tr}[E^*(2)^{-1}]$ are obtained by putting $W = 0$, by replacing $J_{\mu\nu}(p) \to j(p) \delta_{\mu\nu}$ and by differentiating in eqs. (3.16) with respect to $j(p)$. We emphasize this generality because many of the steps of the calculation of $\tilde{Z}$ are the same in all these cases. In order to display this generality clearly we write

$$D(J) = (E - \frac{1}{2} \mathcal{S} - H + i\delta + M(J))$$  \hspace{1cm} (3.18)

where

$$M(J) = iW + J.$$  \hspace{1cm} (3.19)

In sections 4, 5, 6 and parts of section 8, we do not use the form of $M(J)$ and keep it as a general matrix. This will enable the reader to derive other GOE results quite easily.

### 4. Ensemble average and Hubbard–Stratonovitch transformation

In this section, we calculate the ensemble average $\tilde{Z}(E(1), E(2), J)$ of the generating function (3.14), and we simplify the result via the Hubbard–Stratonovitch transformation. From $\tilde{Z}$ the average one- and two-point functions can be calculated using eqs. (3.16). The stochastic variables all reside in the graded matrix $H$, cf. eq. (3.13). We therefore need to calculate the ensemble average of $\exp\{-\frac{1}{2} i(\psi^\dagger \mathcal{L}^{1/2} H \mathcal{L}^{1/2} \psi)\}$ which is easily done using a cumulant expansion. This expansion terminates at second order due to the Gaussian distribution of the elements of $H$. The first moment of the exponent vanishes, and it remains to calculate the second moment, using the definition of $H$ in eq. (3.12b) and the basic property (2.6).

In keeping with our previous notation, we write the components of $\psi$ as $\psi_{\mu\alpha}$ with $\mu = 1, \ldots, N$ and $\alpha = 1, \ldots, 8$. We use the convention that we sum over pairs of equal indices. Explicit calculation shows that

$$\langle \psi^\dagger_{\mu\alpha} L_{\alpha\beta} \psi_{\nu\beta} \rangle = \langle \psi^\dagger_{\mu\alpha} L_{\alpha\beta} \psi_{\nu\beta} \rangle.$$  \hspace{1cm} (4.1)

With the help of this equation, we find that

$$\overline{\frac{1}{4} \langle \psi^\dagger \mathcal{L}^{1/2} H \mathcal{L}^{1/2} \psi \rangle^2} = -\frac{\lambda^2}{2N} \langle \psi^\dagger_{\mu\alpha} L_{\alpha\beta} \psi_{\nu\beta} \rangle \langle \psi^\dagger_{\nu\gamma} L_{\gamma\delta} \psi_{\mu\delta} \rangle.$$  \hspace{1cm} (4.2)

Equation (4.2) can also be expressed in terms of a graded $8 \times 8$ matrix. We define with $1 \leq \alpha, \beta \leq 8$

$$A_{\alpha\beta} = i\lambda \langle L^{1/2} L_{\alpha\gamma} \psi_{\mu\gamma} \psi^\dagger_{\nu\delta} (L^{1/2})_{\delta\beta} \rangle.$$  \hspace{1cm} (4.3)

Then, eq. (4.2) can be written in the form
\[
\frac{i^2}{4} (\psi^* L^{1/2} H L^{1/2} \psi)^2 = \frac{1}{2N} \text{tr} (A^2). \tag{4.4}
\]

This can easily be verified using the definition (A.16) of the graded trace (cf. relations (A.17)).

The form of eq. (4.4) is fundamental for all that follows. The graded matrix \(A_{ab}\) embodies the symmetries of the GOE Hamiltonian; the fact that it has dimension 8 is a consequence of the orthogonal invariance of the GOE and of the fact that we consider the two-point function. (For the one-point function, a 4 \times 4 matrix would suffice.)

Equation (4.4) and the cumulant expansion yield

\[
\exp \left\{ -\frac{i}{2} (\psi^* L^{1/2} H L^{1/2} \psi) \right\} = \exp \left\{ -\frac{1}{4N} \text{tr} (A^2) \right\}. \tag{4.5}
\]

As in the case of the replica trick, we see that calculating the ensemble average, usually the most difficult part in implementing a GOE, is completely straightforward. Moreover, it is independent of the particular problem at hand, i.e., of the form of the matrix \(M(J)\) in eq. (3.19).

The exponent in eq. (4.5) contains a homogeneous polynomial of 4th order in the integration variables \(\psi_{\mu\alpha}\). Therefore, the integral over \(d[\psi]\) is difficult to calculate as it stands. As in the case of the replica trick, we circumvent this difficulty by the Hubbard–Stratonovitch transformation. Since the matrix \(A\) of eq. (4.3) is a graded matrix, the transformation now involves an integration over both commuting and anticommuting variables. The symmetries of the transformation are determined by the symmetries of \(A\). We therefore display the matrix \(B_{ab} = \Sigma_{\mu} \psi_{\mu\alpha} \psi^*_{\mu\beta}\) explicitly in table 4.1. From this matrix, the matrix \(A\) is obtained via eq. (4.3) in a straightforward fashion since \(L\) is diagonal. We briefly discuss the structure of the matrix \(B_{ab}\). The 4 \times 4 block in the upper left-hand corner is a real and symmetric matrix. It is bilinear in the commuting integration variables, and we therefore refer to it as the “Boson–Boson” block. (We deviate here intentionally from the standard nomenclature for reasons of brevity and aesthetics.) The number of independent elements is 10. The 4 \times 4 block in the lower right-hand corner is a Hermitean matrix. [We recall the convention (A.4) for taking the complex conjugate of an anticommuting variable.] It is bilinear in the anticommuting integration variables and is referred to as the “Fermion–Fermion” block. It consists of commuting elements, too. The number of independent real elements is 6. (Note the special symmetry properties among non-diagonal elements.)

The 4 \times 4 blocks in the upper right-hand and the lower left-hand corner are linear in both the

<table>
<thead>
<tr>
<th>Table 4.1</th>
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<tbody>
<tr>
<td>The matrix (B_{ab} = \Sigma_{\mu} \psi_{\mu\alpha} \psi^*_{\mu\beta}) with (1 \leq a, \beta \leq 8). To simplify the notation, we have omitted the summation index (\mu) everywhere.</td>
</tr>
<tr>
<td>(S^{(1)} S^{(1)})</td>
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<tr>
<td>(S^{(1)} S^{(1)})</td>
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<td>(x^{(2)} S^{(1)})</td>
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<td>(x^{*}(2) S^{(1)})</td>
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commuting and the anticommuting integration variables. They consist of anticommuting elements and are referred to as the “Boson–Fermion” blocks. Within each block, the number of independent elements is 16. The upper right-hand block is obtained by transposition and complex conjugation of the lower left-hand block, while the lower left-hand block is obtained from the upper right-hand block by transposition, complex conjugation, and multiplication with minus one. These statements and the definition (A.14) imply that the graded matrix $B$ is Hermitean, $B^\dagger = B$. It has special symmetry properties in the Fermion–Fermion block, and in the Boson–Fermion blocks.

It is often useful to arrange graded $8 \times 8$ matrices differently and in such a way that all integration variables with $p = 1$ ($p = 2$) appear in rows and columns with labels $\alpha, \beta \leq 4$ ($\alpha, \beta \geq 5$, respectively). Labelling rows and columns of the matrix $B$ from 1 to 8, this is achieved by arranging both rows and columns in the sequence (1, 2, 5, 6, 3, 4, 7, 8). We refer to this representation as to the $[1, 2]$ block notation for obvious reasons. In the $[1, 2]$ block notation, the (1, 1) block contains the terms relevant for the calculation of the one-point function [eq. (3.16a)], the (2, 2) block contains the terms relevant for the calculation of the complex conjugate of the one-point function [eq. (3.16b)], while the (1, 2) block and the (2, 1) block contain terms which appear only in the calculation of the two-point function [eq. (3.16c)]. (When calculating graded traces and other expressions in this notation, the operations have, of course, to be modified accordingly.)

To carry out the Hubbard–Stratonovitch transformation, we introduce 16 commuting and 16 anticommuting integration variables. When arranged in matrix form, the resulting matrix $\sigma$ is required to have the same symmetry properties as the matrix $B$, except for the following modification. To make the integral (4.6) convergent, all elements in the Fermion–Fermion block are multiplied by $i$. In Boson–Fermion block notation, we also give $\sigma$ in the latter form in table 4.3 to facilitate our later discussion. The quantities $a, b, c, d, a_{11}, a_{22}, b_{11}, b_{22}, c_{11}, c_{22}$ are 10 real commuting variables. The quantities $z_{11}$ and $z_{22}$ are commuting and real, too, while the quantities $z$ and $w$ are commuting and complex, giving a total of 16 commuting real variables. Greek letters are used to designate the 16 anticommuting variables. While table 4.2 shows that $\sigma$ has essentially the same symmetry properties as $B$, table 4.3 justifies the notation: Quantities with a double index refer to the (1, 1) or to the (2, 2) block, respectively. All commuting variables (including $\text{Re} z$, $\text{Im} z$, $\text{Re} w$, $\text{Im} w$) range from $-\infty$ to $+\infty$; the volume element for a complex variable like $z$ is $\{d(\text{Re} z)d(\text{Im} z)\}$. (It is shown in section 5 that this naive choice of integration contours is not permissible. The steps carried out in the present section have purely formal significance.)

For the anticommuting variables, we write the volume element as $d\eta_{11}^* d\eta_{11} d\rho_{11}^* d\rho_{11} d\eta_{11}^* d\eta_{11} d\rho_{11}^* d\rho_{11} \cdots$. The product of all differentials defines the volume element $d[\sigma]$. We have

<table>
<thead>
<tr>
<th>Table 4.2</th>
<th>Table 4.3</th>
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<tbody>
<tr>
<td>The matrix $\sigma$ in Boson–Fermion block notation</td>
<td>The matrix $\sigma$ in $[1, 2]$ block notation</td>
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<tr>
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<tr>
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</table>
\int d[\sigma] \exp\{-(N/4) \text{trg} \sigma^2\} = +1. \quad (4.6)

This can be verified using the formulas of appendix B. We now shift the integration variables \( \sigma \) in eq. (4.6), using the transformation \( \sigma \to \sigma + (1/N)A \). It is shown in [11] and easily verified that under this transformation, the integral (4.6) does not change its value. Hence,

\[ \exp\{(1/4N) \text{trg} A^2\} = \int d[\sigma] \exp\{-(N/4) \text{trg}(\sigma^2) - \frac{1}{2} \text{trg}(\sigma A)\}. \quad (4.7) \]

Equation (4.7) is the form of the Hubbard–Stratonovitch transformation involving both commuting and anticommuting variables. By direct calculation and using the definition (4.3), we find that

\[ \text{trg}(\sigma A) = i \lambda \psi^\dagger_{\mu \alpha} (L^{1/2})_{\alpha \beta} \sigma_{\beta \gamma} (L^{1/2})_{\gamma \delta} \psi_{\mu \delta}. \quad (4.8) \]

Using eqs. (4.7) and (4.8) in the expression for \( \tilde{Z} \), and formally interchanging the integrations over \( d[\psi] \) and \( d[\sigma] \), we find

\[ \tilde{Z}(E(1), E(2), J) = \int d[\sigma] \exp\left\{-\frac{N}{4} \text{trg}(\sigma^2)\right\} \cdot \int d[\psi] \exp\left\{\frac{i}{2} (\psi^\dagger L^{1/2} N(J) L^{1/2} \psi)\right\} \quad (4.9) \]

where

\[ N(J) = E - \frac{1}{2} \xi + i \delta + M(J) - \lambda \Sigma \quad (4.10) \]

and

\[ \Sigma = \{\delta_{\mu \nu} \sigma_{\alpha \beta}\}. \quad (4.11) \]

The calculation of an integral over an exponential with argument \( \frac{i}{2} (\psi^\dagger L^{1/2} N(J) L^{1/2} \psi) \) is not completely straightforward because the matrix \( \sigma \) is not diagonal. The calculation is carried out in appendix C. It yields

\[ \tilde{Z}(E(1), E(2), J) = \int d[\sigma] \exp\left\{-\frac{N}{4} \text{trg}(\sigma^2) - \frac{1}{2} \text{trg} \ln N(J)\right\}. \quad (4.12) \]

The first of the two traces extends over a graded 8 x 8 matrix, the second one over a graded 8N x 8N matrix. This is indicated by the indices below the trace symbol.

We emphasize that the Hubbard–Stratonovitch transformation (4.7) results in a very substantial simplification of the problem. The 8N integration variables in the original expression for \( Z \), eq. (3.14), have disappeared. We are left with 16 commuting and 16 anticommuting variables, independent of the value of \( N \).

The reader will have noticed that the construction of the matrix \( \sigma \), suggested by the form of the matrix \( B \) in table 4.1, ensures that the number of independent variables in both matrices is the same.
This is one of the conditions which must be met to shift the integration variables as was done in going from eq. (4.6) to eq. (4.7). However, the basic quantity of the theory is not the matrix \( B \), but the matrix \( A \) of eq. (4.3). Indeed, the form of \( A \) is determined by the orthogonal invariance of the GOE and by the fact that we calculate the two-point function. Since \( A = i\lambda L^{1/2} BL^{1/2} \), the matrices \( A \) and \( B \) do not have the same symmetry properties. It appears desirable to model the construction of \( \sigma \) after the matrix \( A \) rather than after the matrix \( B \). However, this is not possible in a straightforward fashion. For instance, for the matrix \( \tau = L^{1/2} \sigma L^{1/2} \) (which is obtained from \( \sigma \) by an operation analogous to the one which leads from \( B \) to \( A \) when \( \sigma \) is chosen according to table 4.2), the expression \( \text{tr}g(\tau^2) \) is not positive definite as would be necessary for the Gaussian integrals in eq. (4.6) to converge. The problem requires extensive discussion. This is the subject of section 5.

5. Hyperbolic symmetry, convergence problems, and the structure of the saddle-point manifold

In this section, we address the question: Is it possible to implement the symmetry of the matrix \( A \) in the integration over \( d[\sigma] \), and how can this be done? It turns out that this question is of central importance to the development of the theory. For the case of the replica trick, i.e., commuting variables only, this question has been thoroughly discussed, and answered, in ref. [2]. New aspects arise, however, due to the appearance of both commuting and anticommuting variables in the present formalism. These lead to the occurrence of both the Boson–Boson block, and the Fermion–Fermion block, in the representation of \( \sigma \) in table 4.2. It will be shown that the implementation of symmetry properties is different for the two cases. Without repeating the detailed discussion of ref. [2] which in the present context is fully applicable to the Boson–Boson block, we believe that an elucidation of the possibilities and problems in both cases is very much called for, as only in this way will the reader be able to appreciate the differences between both cases, and the resulting modifications in the structure of the theory. We proceed in two steps. First, we show in a somewhat simplistic fashion that there is a close relationship between the following three topics: (i) The symmetry of the matrix \( A \), (ii) the mathematical justification of the Hubbard–Stratonovitch transformation, and the convergence of various integrals therein, and (iii) the structure of the saddle-point manifold. This manifold is important for the method of steepest descent, used in implementing the limit \( N \to \infty \). Subsequent to the elucidation of these three points with the help of simple examples, we introduce the parametrization of the \( \sigma \)-integrals which is used in the remainder of this paper. This parametrization differs from the one used by Wegner [15] and is an extension of the parametrization introduced by Pruisken and Schäfer [16].

The present section deals exclusively with properties of the average two-point function. By way of excuse, we devote a few comments to the average one-point function. This latter function can be obtained from the average of the simple expression (3.2). Evaluating the ensemble average and performing the Hubbard–Stratonovitch transformation, one is led to a closed expression. The procedure is free of the problems discussed in the present section. It gives a simple and useful integral representation of \( S_{\mu\nu} \) (or \( (D^{-1})_{\mu\nu} \)) which is valid for every finite \( N \). In the limit \( N \to \infty \), the one-point function can also be calculated via the replica trick. This was done in ref. [10]. No new insight is gained using Grassmann variables in this limit. This is why we focus attention here exclusively on the two-point function.

We turn to the three topics mentioned above: (i) symmetry of the matrix \( A \); (ii) convergence of the \( \psi \)- and \( \sigma \)-integrals in eq. (4.9); (iii) structure of the saddle-point manifold obtained from eq. (4.12). As stated above, these points have, for the case of commuting variables, been thoroughly discussed in ref. [2]. Deferring all details and the complete presentation of the theory to appendices D and E, we aim here at the summary of the main points.
5.1. Symmetry properties

The basic quantity of the theory is the second moment (4.2) or, more precisely, the form

\[ (\psi_{\mu\alpha} L_{\alpha\beta} \psi_{\nu\delta}) = \sum_{p=1}^{2} (-)^{p+1} \left\{ \sum_{i=1}^{2} S_\mu(p) S_\nu(p) + \chi_\mu(p) \chi_\nu(p) + \chi_\mu^*(p) \chi_\nu^*(p) \right\}. \]  

(5.1)

We consider the group \( G \) of linear, non-singular transformations of the original variables \( \{S_\mu(p), \chi_\mu(p), \chi_\mu^*(p)\} \) of integration under which the form (5.1) is invariant. We require that the transformed variables \( \{\tilde{S}_\mu(p), \tilde{\chi}_\mu(p), \tilde{\chi}_\mu^*(p)\} \) also have the properties that the \( \{\tilde{S}_\mu(p)\} \) are real and commuting, that the \( \{\tilde{\chi}_\mu(p), \tilde{\chi}_\mu^*(p)\} \) are anticommuting and that \( \tilde{\chi}_\mu(p)^* = \tilde{\chi}_\mu^*(p) \). The full group \( G \) is constructed in appendix D. Here, we only consider two special cases: A transformation \( \hat{T}_1 \) which leaves the variables \( \{S_\mu(p), \chi_\mu(p), \chi_\mu^*(p)\} \) unchanged and transforms the \( \{S_\mu^1(p)\} \) according to

\[
\begin{pmatrix}
S_\mu^1(1) \\
S_\mu^1(2)
\end{pmatrix} \longrightarrow \begin{pmatrix}
\cosh \beta & \sinh \beta \\
\sinh \beta & \cosh \beta
\end{pmatrix} \begin{pmatrix}
S_\mu^1(1) \\
S_\mu^1(2)
\end{pmatrix}, \quad \text{all } \mu ; \tag{5.2}
\]

and a transformation \( \hat{T}_2 \) which leaves the variables \( \{S_\mu^2(p)\} \) unchanged and transforms the anticommuting variables according to

\[
\begin{pmatrix}
\chi_\mu(1) \\
\chi_\mu(2)
\end{pmatrix} \longrightarrow \begin{pmatrix}
\cosh \beta & \sinh \beta \\
\sinh \beta & \cosh \beta
\end{pmatrix} \begin{pmatrix}
\chi_\mu(1) \\
\chi_\mu(2)
\end{pmatrix}, \quad \text{all } \mu . \tag{5.3}
\]

The complex conjugate variables \( \{\chi_\mu^*(p)\} \) also transform according to (5.3) as is seen by complex conjugation. In the transformations (5.2) and (5.3), the variable \( \beta \) is real with \( -\infty < \beta < \infty \). It is straightforward to check that the form (5.1) is invariant under both \( \hat{T}_1(\beta) \) and \( \hat{T}_2(\beta) \) owing to the “indefinite metric” caused by the factor \( (-)^{p+1} \). (Hence the name “hyperbolic symmetry” introduced by Schäfer and Wegner [2].

Any transformation \( \hat{T} \) among the variables \( \{S_\mu^i(p), \chi_\mu(p), \chi_\mu^*(p)\} \) induces a transformation \( T \) of the matrix \( A \) in eq. (4.3), with

\[
A \Rightarrow T A T^{-1} \quad \text{and} \quad T = L^{1/2} \hat{T} L^{-1/2}; \quad T^{-1} = L^{-1/2} \hat{T}^* L^{1/2}. \tag{5.4}
\]

In both cases of the relations (5.2) and (5.3), the matrix \( T \) is the unit matrix except for four elements, given by

\[
\begin{pmatrix}
\cosh \beta & -i \sinh \beta \\
i \sinh \beta & \cosh \beta
\end{pmatrix}. \tag{5.5}
\]

Thus, \( T \) is complex orthogonal, \( T^{-1} = T^* \). The sets of matrices \( \{T_1(\beta)\} \) and \( \{T_2(\beta)\} \) each form a non-compact \( (-\infty < \beta < +\infty) \) group of transformations.

Since \( A \) enters the Hubbard–Stratonovitch transformation (4.7) in the form \( \text{trg}(\sigma A) \), the transformation (5.4) is equivalent to

\[
\sigma \Rightarrow T^{-1} \sigma T. \tag{5.6}
\]

A full implementation of the symmetry defined by eq. (5.6) requires \( \sigma \) to be parametrized in such a
fashion that the transformations $T$ can be absorbed into $\sigma$, without affecting either the form of $\sigma$, or its domain of integration. For the special matrices $T_1(\beta)$ and $T_2(\beta)$ introduced above, we now show that $\sigma$ is not invariant under the transformation (5.6) when the parametrization given in table 4.2 is used. For the transformation $T_1(\beta)$, the only variables affected in the Boson–Boson block $\sigma_B$ of $\sigma$ are $a_{11}, a_{22}$ and $a$ (see table 4.2); $T_1(\beta)$ leaves all the variables in the Fermion–Fermion block $\sigma_F$ of $\sigma$ unchanged. For the transformation $T_2(\beta)$, we note that $\sigma_B$ remains unchanged. Concerning $\sigma_F$, we note that the $2 \times 2$ matrix 
\[
\begin{pmatrix}
-w & y^* \\
-y & w^*
\end{pmatrix}
\] 
appearing in table 4.2 can be written as a special unitary $2 \times 2$ matrix $U$ times a real positive factor $\mu$. (Details are given in section 5.3 below.) After a little algebra we find that $T_2(\beta)$ affects $\sigma_F$ via the transformation 
\[
\begin{pmatrix}
(z_{11} & \mu \\
\mu & z_{22}
\end{pmatrix} \rightarrow \begin{pmatrix}
\cosh \beta & i \sinh \beta \\
-i \sinh \beta & \cosh \beta
\end{pmatrix} \begin{pmatrix}
(z_{11} & \mu \\
\mu & z_{22}
\end{pmatrix} \begin{pmatrix}
\cosh \beta & -i \sinh \beta \\
\sinh \beta & \cosh \beta
\end{pmatrix}
\] 
(5.7)

The effect of $T_1(\beta)$ on $\sigma_B$ is similar, with $(z_{11}, z_{22}, \mu)$ in the relation (5.7) replaced by $(a_{11}, a_{22}, a)$, respectively. The symmetry of the theory would be implemented if the transformation (5.7) (which acts on a real and symmetric matrix) were again to yield a real and symmetric matrix. This is not the case. (The off-diagonal elements are purely imaginary.) It follows that the parametrization of $\sigma$ in table 4.2 is not consistent with the symmetry of the form (5.1).

5.2. Convergence questions

Is it possible to implement the symmetry of the form (5.1) in the parametrization of the composite variables $\sigma$? This question cannot be answered without discussing the convergence properties of the integrals appearing in eqs. (4.7), (4.9) and (4.12). We begin with the following observation.

The function $\tilde{Z}(E(1), E(2), J)$ is not invariant under transformations $\tilde{T}$ which leave the form (5.1) invariant. Indeed, the terms $\mathcal{E}, \delta$ and $M(J)$ in eqs. (4.9) and (4.10) break this symmetry. For this reason, we now consider $\tilde{Z}(E(1), E(2), J)$ at $M(J) = 0$ and $\mathcal{E} = 0$ (or $E(1) = E(2)$) and denote this function by $Z(E)$, anticipating that symmetries of $Z(E)$ will not be those of the full $\tilde{Z}(E(1), E(2), J)$. We keep the infinitesimal matrix $\delta$ to ensure convergence.

By virtue of the normalization condition, we have $\lim_{\eta \to 0} \tilde{Z}(E) = 1$. We note, however, that putting $\eta = 0$ (or $\delta = 0$) prior to calculating the integrals (3.14) with (4.2), yields a divergent result. Indeed, the integrand is now invariant under the transformations $\tilde{T}$. These form a non-compact group. Introducing as new integration variables (among others) the group parameter $\beta$ (on which the integrand does not depend), we obtain a divergent integral. While the normalization condition $\tilde{Z}(E) = 1$ hides this singularity in the expression $\lim_{\eta \to 0} \tilde{Z}(E)$, the singularity becomes apparent when one considers derivatives of $\tilde{Z}(E(1), E(2), J)$ at $M(J) = 0 = \mathcal{E}$ as in eq. (3.16c). (This, in fact, is the reason why the two-point function contains information about fluctuation properties of the system which are of the order of the mean level spacing $d$.) These facts suggest that it is very desirable to implement the symmetry of the form (5.1) in the parametrization of the set of composite variables $\sigma$. Can this be done?

In writing eq. (4.9), we have interchanged the integrations over $d[\sigma]$ and $d[\psi]$. This is justified if the $\psi$-integrals converge, and if the $\sigma$-integrals converge uniformly in the $\psi$-variables. Both conditions are trivially fulfilled for those $\psi$-variables which are anticommuting. Since the variables in $\sigma_F$ appear in eq. (4.9) only in conjunction with anticommuting $\psi$-variables, uniform convergence is no problem, and since all integrals involving anticommuting composite variables are trivial anyway, it is only the (commuting) variables in $\sigma_B$ for which the operations leading to eq. (4.9) may not be legitimate.
It is easy to see that for some variables in $\sigma_B$, the integrals indeed do not converge uniformly, and the corresponding $S_\mu(\sigma)$-integrals diverge. Consider, for instance, the integration over the variable $a$ (see table 4.2) in eq. (4.9). The integrand is the exponential of $-\frac{N}{2}a^2 + a \sum_{\mu} S_\mu(1) S_\mu(2)$. This integral does not converge uniformly in $\sum_{\mu} S_\mu(1) S_\mu(2)$: A cutoff in the $a$-integration that guarantees a good approximation will have to depend on the value of $\sum_{\mu} S_\mu(1) S_\mu(2)$. By the same token, the presence of the term $a \sum_{\mu} S_\mu(1) S_\mu(2)$ in the exponent jeopardizes the convergence of the integrals over $S_\mu(1)$ and $S_\mu(2)$. Indeed, these integrations are only convergent (prior to the introduction of the Hubbard–Stratonovitch transformation) by virtue of the convergence-generating infinitesimal factor $\eta$. However, the positive-definiteness of the form $\eta \cdot \sum_{\mu,\nu} (S_\mu(\sigma))^2$ is destroyed when the term $a \sum_{\mu} S_\mu(1) S_\mu(2)$ with arbitrary real $a$ is added to it.

This difficulty is overcome, and the procedure leading to eq. (4.9) is made legitimate, if we implement the full symmetry of the form (5.1) in the parametrization of $\sigma_B$. Deferring the full description to appendices D and E, we here elucidate this statement, taking the transformations $T_1(\beta)$ as an example. The form of the relevant $2 \times 2$ matrix in (5.5) suggests that we replace the matrix $(a_{11} \quad a_{22})$ in $\sigma_B$ by

$$
\begin{pmatrix}
\cosh \beta & i \sinh \beta \\
-i \sinh \beta & \cosh \beta
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & 0 \\
0 & \sigma_2
\end{pmatrix}
\begin{pmatrix}
\cosh \beta & -i \sinh \beta \\
i \sinh \beta & \cosh \beta
\end{pmatrix},
$$

introducing as integration variables the real quantities $\sigma_1$, $\sigma_2$ and $\beta$ instead of $a_{11}$, $a_{22}$ and $a$. It is clear that the form (5.8) has the full symmetry of the group of transformations $\{T_1(\beta)\}$. Indeed, transforming the expression (5.8) with a matrix $T_1(\beta)$ we retrieve the form (5.8) with $\beta$ replaced by $\beta + \beta$. This change can be absorbed by a shift of integration variables.

It is most gratifying to see that by introducing the parametrization (5.8), we also overcome the problems explained above of non-uniform convergence in $\sigma$, and lack of convergence in $S_\mu(\sigma)$. Indeed, the form (5.8) amounts to replacing the variable $a$ by the imaginary quantity $-i \sinh \beta \cosh \beta (\sigma_1 - \sigma_2)$, and since

$$
\left| \exp \left\{ i \sinh \beta \cosh \beta (\sigma_2 - \sigma_1) \sum_{\mu} (S_\mu(1) S_\mu(2)) \right\} \right| = 1,
$$

the integration over the new variables $\sigma_1$, $\sigma_2$ and $\beta$ converges uniformly in the $\psi$-variables, and integration over the latter is guaranteed to be convergent.

But is the integral over the new variables $\sigma_1$, $\sigma_2$ and $\beta$ convergent at all? The expression $\text{tr} \sigma^2$ (which guarantees convergence for the parametrization of table 4.2) changes as we introduce the form (5.8); the relevant part of $\text{tr} \sigma^2$ relating to $\sigma_1$, $\sigma_2$ and $\beta$ is given by $(\sigma_1^2 + \sigma_2^2)$. While the integrals over $\sigma_1$ and $\sigma_2$ are convergent as before, we see that there is no term guaranteeing convergence of the $\beta$-integration. This difficulty can be overcome very easily, however, by adding in the exponent an infinitesimal symmetry-breaking term

$$
-i \frac{N}{2} \alpha \text{tr} \text{g}(\sigma_L_B) = -i \frac{N}{2} \alpha \text{tr} \text{g}(\sigma_B L_B)
$$

with $\alpha > 0$ infinitesimal, and $L_B$ the projection of the matrix $L$ defined in section 3 onto the Boson–Boson part. For the case of the parametrization (5.8), this term reads $-i (N/2) \alpha (\sigma_1 - \sigma_2) (\cosh^2 \beta + \sinh^2 \beta)$. By choosing the contour of the integrals over $\sigma_1$ and $\sigma_2$ in such a way that
\[
\sigma_1 \rightarrow \sigma_1 - i\gamma, \quad \sigma_2 \rightarrow \sigma_2 + i\gamma, \quad \gamma > 0
\] (5.11)

we generate in the exponent the expression

\[-\alpha \gamma N(\cosh^2 \beta + \sinh^2 \beta),\] (5.12)

which guarantees convergence. We emphasize that the choice (5.11) is consistent with the deformation of the contours needed to reach the saddle points, see the discussion given below.

We have shown that the parametrization of \(\sigma_B\) in table 4.2 is not satisfactory as it stands: The integrals do not all converge in the \(\psi\)-variables, and \(\sigma_B\) does not have the symmetry of the form (5.1). As was indicated, both problems can be overcome together by choosing a new parametrization in terms of a non-compact group.

The situation is very different for the variables in \(\sigma_F\). Although these, too, do not accommodate the symmetry properties of the form (5.1), there are no convergence problems as in the case of \(\sigma_B\) that would force us to adopt a parametrization different from that of table 4.2. It may thus appear that the implementation of symmetry in \(\sigma_F\) is a matter of choice, not of necessity as for \(\sigma_B\). This, however, is not the case. It turns out that it is actually impossible to implement the symmetry of the form (5.1) in \(\sigma_F\), and that it is necessary to keep the parametrization of table 4.2.

To see this, we recall the arguments leading to the transformation (5.7), cf. also section 5.3. Putting for simplicity the unitary transformation \(U = 1\), and proceeding in analogy to the parametrization (5.8), we are led to consider instead of the 2 x 2 matrix \((\begin{smallmatrix} U_1 \amp U_2 \\ U_3 \amp U_4 \end{smallmatrix})\) the matrix

\[
\begin{pmatrix}
\cosh \beta & i \sinh \beta \\
-i \sinh \beta & \cosh \beta
\end{pmatrix}
\begin{pmatrix}
\tau_1 & 0 \\
0 & \tau_2
\end{pmatrix}
\begin{pmatrix}
i \sinh \beta & -i \sinh \beta \\
-\sinh \beta & \cosh \beta
\end{pmatrix}
\] (5.13)

with \(\tau_1\), \(\tau_2\) and \(\beta\) as the new real variables of integration. Convergence problems arise as in the case of \(\sigma_B\) because \(\text{trg} \sigma^2\) contains only \((\tau_1^2 + \tau_2^2)\) and not \(\beta\). This time, however—and this is the central difference—convergence cannot be enforced by adding in the exponent a term analogous to the expression (5.10). This is because \(\sigma_F\) contains an overall factor \(i\) (see table 4.2) which is necessary to make \(\text{trg} \sigma^2\) positive definite in \(\sigma_F\) and which cannot be abandoned. This additional factor \(i\) causes the argument in the exponent (which corresponds to the expression (5.12)) to be imaginary. Adding a term like expression (5.12) in the \(\sigma_F\)-variables is therefore not helpful in producing convergence. One might consider adding in the exponent instead of the term just considered a term

\[
+\alpha \frac{N}{2} \text{trg}(\sigma L_F) = +\alpha \frac{N}{2} \text{trg}(\sigma_F L_F)
\] (5.14)

with \(\alpha\) infinitesimal and positive, and \(L_F\) the projection of \(L\) onto the Fermion–Fermion block. With the replacement

\[
\tau_1 \rightarrow \tau_1 - i\gamma, \quad \tau_2 \rightarrow \tau_2 + i\gamma, \quad \gamma > 0
\] (5.15)

in the parametrization (5.13), this yields in the exponent the term \(-\alpha \gamma N(\cosh^2 \beta + \sinh^2 \beta)\) which guarantees convergence as before. However, the problem is only hidden in this way because new difficulties arise when we restrict \(\sigma\) to the saddle points. These difficulties (which do not occur for \(\sigma_B\))
are described below. Moreover, the combination of expressions (5.10) and (5.14) would break the graded symmetry by treating $\sigma_B$ and $\sigma_F$ differently. We conclude that we cannot implement the symmetry of the form (5.1) in the Fermion–Fermion block and keep the integrals convergent, and that we are forced to use the original parametrization for $\sigma_F$ as given in table 4.2.

To enable the reader to appreciate the results of sections 5.3 and 5.5 below, we now display a particular way of viewing the parametrization of $\sigma_F$ in table 4.2. We recall that the transformation (5.7), applied to the real and symmetric $2 \times 2$ matrix $\left( \begin{array}{cc} \mu & z_{12} \\ \mu & z_{22} \end{array} \right)$, does not yield a matrix with the same symmetry properties. If, however, we replace in the transformation (5.7) the variable $\beta$ by $i\hat{\beta}$, and thereby change the complex orthogonal transformation into a real orthogonal one, then the transformed matrix is indeed again real and symmetric. In other words, taking instead of the parametrization (5.13) the choice

$$
\begin{pmatrix}
  z_{11} & \mu \\
  \mu & z_{22}
\end{pmatrix} \rightarrow \begin{pmatrix}
  \cos \hat{\beta} & -\sin \hat{\beta} \\
  \sin \hat{\beta} & \cos \hat{\beta}
\end{pmatrix} \begin{pmatrix}
  \tau_1 & 0 \\
  0 & \tau_2
\end{pmatrix} \begin{pmatrix}
  \cos \hat{\beta} & \sin \hat{\beta} \\
  -\sin \hat{\beta} & \cos \hat{\beta}
\end{pmatrix}
$$

is consistent with the symmetry properties of the matrix $\sigma_F$ in table 4.2. In effect this amounts to a “compactification”: The real variable $\beta$ with $-\infty < \beta < +\infty$ is replaced by $i\hat{\beta}$ with $0 \leq \hat{\beta} \leq \pi$, and the non-compact group of complex orthogonal transformations is replaced by the compact real orthogonal group. This is why the $\sigma_F$-integrals converge as they stand, and this is ultimately the reason why the saddle-point manifold derived from $\sigma_F$ is compact while that derived from $\sigma_B$ (after implementation of the parametrization (5.8)) is not.

In conclusion, we have seen that convergence arguments, combined with the anticipated saddle-point properties, leave us no choice but to adopt a non-compact parametrization for $\sigma_B$, and a compact parametrization for $\sigma_F$. By this we mean that the parametrization for $\sigma_B$ [for $\sigma_F$] is invariant under the non-compact group of transformations (5.8) [under the compact group of transformations (5.16), respectively].

5.3. The saddle points

The saddle points are determined by the condition that the form in the exponent of eq. (4.12) be stationary (vanishing of linear terms in a Taylor expansion). As in the case of the integrals discussed above, we adopt a simplistic view and consider separately the saddle-point equations for $\sigma_B$, and for $\sigma_F$, without explicit reference to the anticommuting variables. Again, we put $\vec{\mathcal{E}} = 0 = M(J)$ to have the case of maximum symmetry.

For $\sigma_B$, the saddle-point equation reads ($\eta \to 0$)

$$
\sigma_B(E - \lambda \sigma_B) = \lambda ,
$$

with $E$ defined in eq. (3.12a). For ordinary variables (rather than matrices), eq. (5.17) has the two solutions

$$
\sigma_0 = \frac{E}{2\lambda} - i\Delta_0 \quad \text{and} \quad \sigma_0^* = \frac{E}{2\lambda} + i\Delta_0
$$

where

$$
\Delta_0 = \left( 1 - \left( E/2\lambda \right)^2 \right)^{1/2}.
$$
The definition (2.6) causes the spectrum of $H$ for $N \to \infty$ (more precisely, the discontinuity of $\text{tr}((E-H)^{-1})$ for $N \to \infty$) to extend from $-2\lambda$ to $+2\lambda$. We are interested only in energies in this domain, $|E| < 2\lambda$. The saddle points (5.18) must be reached by deformation of the original contours of integration. Because of the appearance of the term $i\delta$ under the logarithm [cf. eqs. (4.12), (4.10) and (3.12b)], this is possible only if we choose

$$\sigma^0_B = \begin{pmatrix} \sigma_0 & \sigma_0^* & \emptyset \\ \emptyset & \sigma_0^* & \emptyset \\ \sigma_0 & \emptyset & \sigma_0^* \end{pmatrix}$$

(5.20)

for the saddle-point solution of $\sigma_B$.

Returning to eq. (5.17), we see that this equation is invariant under the symmetry transformations $T_1$. This implies that with $\sigma^0_B$, also $T_1^{-1}\sigma^0_B T_1$ is a saddle point. Using the transformations (5.5) as an example, we find that $T_1^{-1}\sigma^0_B T_1$ differs from $\sigma^0_B$. It follows that eq. (5.17) has a manifold of solutions. Owing to the non-compactness of the group $G$, this manifold is not compact: Evaluating the integrals in eq. (4.12) in the saddle-point approximation for $\mathcal{Z} = 0 = \mathbf{M}(\mathcal{U})$, $\eta = 0$, we obtain a divergent integral. This reflects the divergence in $Z(E)$ discussed above, and is a direct consequence of the fact that $\sigma_B$ has the same symmetry as the form (5.1). Returning to the term $-\frac{1}{2}iN\alpha \text{tr}g(\sigma_B L_B)$ introduced in expression (5.10), and replacing $\sigma_1$ and $\sigma_2$ by their saddle-point values, we obtain the exponential of $\{-N\alpha \Delta_0(\cosh^2 \beta + \sinh^2 \beta)\}$. This is satisfactory since this term is supposed to produce convergence in the $\beta$-integrals. The result is not surprising because the location of the saddle points (5.20) is consistent with the deformation of the integration contour proposed in expression (5.11).

We turn to the saddle points for $\sigma_F$. These are calculated in some detail in order to show the difference between the non-compact parametrization for $\sigma_B$, and the present case.

We observe that with $\mu = \sqrt{(z^2 + |w|^2)^{1/2}}$, we have

$$\begin{pmatrix} z \\ -w \\ z^* \end{pmatrix} = \mu U$$

(5.21a)

where $U$ is special unitary, $UU^\dagger = 1$ and $\det U = 1$. Moreover,

$$\begin{pmatrix} z^* \\ -w^* \\ z \end{pmatrix} = \mu U^\dagger.$$

(5.21b)

This shows that $\sigma_F$ can be written as (see table 4.2)

$$\sigma_F = \begin{pmatrix} U & \emptyset & iz_{11} \cdot 1 & i\mu \cdot 1 \\ \emptyset & 1 & i\mu \cdot 1 & iz_{22} \cdot 1 \\ iz_{11} \cdot 1 & iz_{22} \cdot 1 & U^\dagger & \emptyset \\ \emptyset & 1 & \emptyset & 1 \end{pmatrix}$$

(5.22)

where $1$ stands for the $2 \times 2$ unit matrix. The Fermion–Fermion part of the Lagrangian in eq. (4.12) is given by (we recall that $\mathcal{Z} = 0 = \mathbf{M}$, $\delta = 0$)

$$\mathcal{L}_{FF} = -\frac{N}{4} \text{tr}g[(\sigma_F)^2] - \frac{N}{2} \text{tr}g \ln(E - \lambda \sigma_F).$$

(5.23)

With the parametrization (5.22), this reduces to
Using the parametrization (5.16), we are left with
\[
\mathcal{L}_{\text{FF}} = -\frac{N}{2} \left( \tau_1^2 + \tau_2^2 \right) + N \ln[(E - i\lambda \tau_1)(E - i\lambda \tau_2)].
\]  
(5.25)

The saddle points are obtained by variation of the expression (5.25) with respect to \( \tau_1 \) and \( \tau_2 \). This yields
\[
\tau = \frac{\lambda}{E - i\lambda \tau} \quad \text{or} \quad \tau = \frac{E}{2\lambda} + \lambda_0,
\]  
(5.26)

where the signs are chosen opposite for \( \tau_1 \) and \( \tau_2 \). (We will justify this choice of sign in section 5.5.) Choosing the negative sign for \( \tau \) and transforming back with the help of the transformation (5.16), we obtain
\[
iz_{11} = \frac{E}{2\lambda} - \lambda_0 \cos 2\hat{\beta}, \quad iz_{22} = \frac{E}{2\lambda} + \lambda_0 \cos 2\hat{\beta}, \quad \mu = -\lambda_0 \sin 2\hat{\beta}
\]  
(5.27)

and \( 0 \leq \hat{\beta} \leq \pi \). (If we restrict \( \mu \) to positive values, the domain of \( \hat{\beta} \) must obviously be restricted further.) The values of \( z \) and \( w \) are given by eqs. (5.21) with \( U \) arbitrary but special unitary. The manifold (5.27), (5.21) of saddle points is manifestly compact (we recall \( 0 \leq \hat{\beta} \leq \pi \) and the fact that the \( U \) form a compact group). This also shows the connection with the arguments in section 5.2, and is a clear consequence of the parametrization in table 4.2.

Equation (5.26) shows that, aside from a multiple of the unit matrix, the saddle points for \( \sigma_F \) are \textit{real} in \( \tau \). This is the ultimate reason why a non-compact parametrization of \( \sigma_F \) is impossible: The location of the two saddle points in eq. (5.26) is \textit{inconsistent} with the deformation of the contours of integration proposed in expression (5.15). Put differently, using a non-compact parametrization for \( \sigma_F \) in the exponential of expression (5.14), we obtain an oscillatory function and not a convergence-generating factor.

5.4. \textit{Does the Fermion–Fermion block} \( \sigma_F \) \textit{really have hyperbolic symmetry?}

Summarizing the results of sections 5.1 to 5.3, we state that the combination of convergence arguments for the Hubbard–Stratonovitch transformation and of the structure of the saddle points, forces us to use a non-compact parametrization for \( \sigma_B \), and a compact parametrization for \( \sigma_F \), although the symmetry of the form (5.1) would suggest a non-compact parametrization for both \( \sigma_B \) and \( \sigma_F \). This result may seem unsatisfactory in that we are forced by convergence arguments to break a symmetry (in \( \sigma_F \)) which appears to be basic to the theory. In the present section, we suggest that this is not the case. (The detailed arguments are given in appendix E.) While the symmetry of the form (5.1) is completely determined for the \textit{commuting} \( \psi \)-variables (and, therefore, for \( \sigma_B \)), we do have a freedom of choice between a definite and an indefinite metric in the case of the anticommuting \( \psi \)-variables (and, therefore, \( \sigma_F \)).
This freedom of choice originates from a profound physical difference between the commuting and anticommuting $\psi$-variables. As shown in appendix B and section 3, integration over the commuting $\psi$-variables generates, in the limit $\varepsilon = 0 = W_{\mu\nu}$, a factor $[\det(E - H)]^{-1}$. This expression is singular on the real axis, and it therefore matters greatly whether the real axis is approached from below, or from above. For the two-point function, we combine two expressions in which the real axis is approached from opposite sides. This reflects itself in the convergence-generating factor $\eta$ in the exponent and results in the hyperbolic symmetry of the form (5.1) with respect to the commuting $\psi$-variables. This symmetry leaves us no choice but to adopt a non-compact parametrization for $\sigma_B$.

Not so for the anticommuting $\psi$-variables! Indeed, for $\varepsilon = 0 = W_{\mu\nu}$ integration over the $(\chi_\mu, \chi^*_\mu)$ yields the factor $\det(E - H)$ which is not singular on the real axis: We obtain the same result irrespective of the direction in which the real axis is approached. This is reflected in the fact that there is no need for a convergence-generating factor $\eta$ in the integration over $(\chi_\mu, \chi^*_\mu)$; such integrals always converge. Moreover, there is a twofold freedom of choice in defining the integrals involving the $(\chi_\mu, \chi^*_\mu)$. First, the factor $i$ multiplying the matrix $A$ in eq. (B.3) can be replaced by any complex number; the “metric” of the form (5.1) with respect to the anticommuting variables can thus be changed ad libitum. Second, we are free to interchange the roles of $\chi_\mu$ and $\chi^*_\mu$ without affecting at all the structure of the theory, although this interchange does affect the form of the matrix $C$ in eq. (D.4). We see that both the “metric” of the form (5.1), and the definition of the matrix $C$, are at our disposal as far as the anticommuting variables are concerned.

Appendices D and E describe two different routes to the same parametrization of $\sigma$. In appendix D, we use the symmetry of the form (5.1) as it stands and show that it yields a non-compact parametrization for both $\sigma_B$ and $\sigma_F$. We subsequently “compactify” this parametrization with respect to $\sigma_F$. This procedure is informative in that it bears close analogy to the arguments in sections 5.2 and 5.3. It remains unsatisfactory, however, since “compactification” appears to be an ad-hoc procedure disconnected from the symmetry underlying the basic form (5.1). In appendix E, we therefore give a second derivation, which we start using the above-mentioned freedom of choice in such a way as to arrive at the desired answer. With this derivation, the theory is closed and entirely free of ad-hoc procedures.

5.5. Parametrization of $\sigma$

The singularity of the $\psi$-integrals and, with proper parametrization of $\sigma$, of the $\sigma$-integrals at $\varepsilon = 0 = W_{\mu\nu}, J = 0$ and $\eta = 0$, is a general feature of the GOE. It occurs in every problem involving a GOE Hamiltonian*. The particular GOE symmetry causing the singularity is broken by terms proportional to $\varepsilon$, to $J$, and to $W_{\mu\nu}$. In introducing a parametrization for $\sigma$ which incorporates the above-mentioned singularity, we emphasize the GOE symmetry underlying the problem, and we develop a formulation of the theory which is applicable to a variety of specific physical problems.

This parametrization of $\sigma$ is determined by the arguments presented in sections 5.1 to 5.4, and in appendices D and E. Linear transformations $\hat{T}$ of the original variables $\psi$ of integration induce transformations $\sigma \rightarrow T^{-1}\sigma T$ of the composite variables $\sigma$. The form of $T$ is completely determined by arguments of symmetry and convergence. The transformations obey eqs. (D.17) and can be written in the form

---

* The singularity is, in fact, not a specific GOE property and occurs in the two-point function of many physical systems.
Here, \( R \) is block diagonal in \([1, 2]\) block notation, each block separately obeying eqs. (D.17). (We suppress the index \( c \) which was introduced for clarity in appendix D.) The matrix \( T_0 \) has the form (we use \([1, 2]\) block notation)

\[
T_0 = \begin{pmatrix}
(1 + t_{12} t_{21})^{1/2} & i t_{12} \\
- i t_{21} & (1 + t_{21} t_{12})^{1/2}
\end{pmatrix}.
\]

The matrices \( t_{12} \) and \( t_{21} \) are given in table D.3.

We write the general matrix \( \sigma \) in the form

\[
\sigma = T^{-1} \sigma_D T,
\]

where \( \sigma_D \) is diagonal with diagonal elements \((\sigma_1, \sigma_2, i\sigma_3, i\sigma_3, \sigma_4, i\sigma_5, i\sigma_6, i\sigma_6)\) and \( \sigma_k \) real and commuting, \( k = 1, \ldots, 6 \). The factors \( i \) are introduced to ensure convergence of the Gaussian integrals below. This is the same modification as in table 4.2. The third and fourth, and the seventh and eighth, diagonal elements are required to be equal by reality arguments, see eq. (D.6).

There are 16 commuting and 16 anticommuting independent variables in \( \sigma \), the same number as in the original parametrization of table 4.2. Indeed, for the anticommuting variables, we have 8 independent variables in \( t_{12} \), 4 in \( R_1 \), and 4 in \( R_2 \), cf. tables D.1, D.3 and E.1. For the commuting variables, we have 8 in \( t_{12} \), 6 in \( \sigma_D \) and 4 each in \( R_1 \) and \( R_2 \). Among the variables in \( R_1 \) and \( R_2 \), however, the variables forming the \( 2 \times 2 \) blocks in the lower right-hand corner (see tables D.1 and E.1) should not be counted since they transform the diagonal \( 2 \times 2 \) matrices \((0_{10} i_{10})\) and \((0_{10} i_{10})\) into themselves, leaving us with a total of 16 commuting variables.

With \( \sigma \) given by eq. (5.30), we now consider the integral

\[
\int d[\sigma] \exp \left\{ - \frac{N}{4} \text{trg} \sigma^2 - i \frac{N}{2} \alpha \text{trg}(\sigma L) \right\}.
\]

The last term in the exponent corresponds to expression (5.10) with \( \alpha \) positive and infinitesimal. It has been added to ensure convergence. To preserve graded symmetry, we have replaced \( \sigma_B L_B \) by \( \sigma L \). Due to the compact parametrization introduced for \( \sigma_f \), this is permissible. To calculate this term explicitly, we note that \((R^{-1} \sigma_D R)\) is block diagonal; we write

\[
(R^{-1} \sigma_D R) = \begin{pmatrix}
P_{11} & 0 \\
0 & P_{22}
\end{pmatrix}.
\]

Then, explicit calculation shows that

\[
\text{trg}(\sigma L) = \text{trg}[P_{11}(1 + 2t_{12} t_{21})] - \text{trg}[P_{22}(1 + 2t_{21} t_{12})],
\]

where we have used that \( T_0^{-1} = L^{-1} T_0^* L \). Convergence is assured if the diagonal Boson–Boson parts of \( P_{11} \) and \( P_{22} \) are given infinitesimal imaginary parts,
\[ P_{11}^0 \rightarrow P_{11}^0 - i \gamma 1, \quad P_{22}^0 \rightarrow P_{22}^0 + i \gamma 1, \quad \gamma > 0. \]  

This can be achieved by a suitable choice of \( \sigma_D \). For the parametrization (5.30), the volume element is non-trivial and must be worked out. It is given by the Jacobian of the transformation (5.30), see appendix F. Introducing the independent elements of \( P_{11}, P_{22} \) and of \( t_{12} \) as variables of integration, we have

\[
d[\sigma] = \mathcal{F}(P) d[P] d\mu(t). \tag{5.35}
\]

Here, \( \mathcal{F}(P) \) depends only on the eigenvalues of \( P_{11} \) and \( P_{22} \), see appendix F, and \( d\mu(t) \) only on the “eigenvalues” of \( t_{12} \), and the differentials of the variables in \( t_{12} \), see appendices I and K.

The parametrization (5.30) and the formula (5.35) leave us the choice of using the variables in \( P_{pp} \) and \( t_{12} \), or the independent elements of \( \sigma \), as variables of integration. In the latter case, the integral (5.31) attains nearly the same form as eq. (4.6), although it differs by the presence of the convergence-generating factor, and in that the paths of integration must now be taken in conformity with the range of variables in \( P_{pp} \) and \( t_{12} \) defining the parametrization (5.30). The integrals over the commuting variables in \( \sigma \) extend over an infinite domain, and the shift \( \sigma \rightarrow \sigma + (1/N)A \) used in going from eq. (4.6) to eq. (4.12) is justified as it amounts only to a finite displacement of the contours of integration. This is best seen if we assume that \( A \) can be diagonalised by a matrix \( T \) introduced in eqs. (D.12),

\[
A = T A_D T^{-1}, \tag{5.36}
\]

with \( A_D \) diagonal. (We show at the end of appendix I that such diagonalisation is always possible.) The transformation \( T \) can be absorbed into \( \sigma \), so that \( \sigma \rightarrow T^{-1} \sigma T \) defines the new variables of integration and the shift \( \sigma \rightarrow \sigma + (1/N)A \) reduces to a shift \( \sigma_D \rightarrow \sigma_D + (1/N)A_D \) of the diagonal elements which poses no problems. On the r.h.s. of eq. (4.7) this shift yields in the exponent the additional term \(-\frac{1}{2} \alpha \lambda \sum_{\mu \nu} (S_\mu'(p))^2\), amounting to the replacement \( \eta \rightarrow \eta + \alpha \lambda \) which is permissible since \( \alpha \lambda > 0 \).

In this way, the steps leading to eq. (4.12) have eventually become mathematically rigorous. Equation (4.12) is fully justified for the parametrization (5.30) provided the integral (5.31) is normalized to unity. The normalization is not discussed here; this point is deferred until section 8.

The saddle points are obtained by variation of the effective Lagrangian, taken at \( \varepsilon = 0 = W_{\mu \alpha} \),

\[
-\frac{N}{4} \text{tr} \sigma^2 - \frac{N}{2} \text{tr} \ln(E - \lambda \sigma). \tag{5.37}
\]

Using the form (5.30) we notice that expression (5.37) depends only on the diagonal elements \( \sigma_1, \ldots, \sigma_6 \). For the saddle point \( \sigma_D^0 \), we find

\[
\sigma_D^0 = \frac{E}{2\lambda} - i \Delta_0 L. \tag{5.38}
\]

In the vicinity of the saddle point, we use eq. (5.32) and write

\[
R^{-1} \sigma_D R = \sigma_D^0 + \delta P \tag{5.39}
\]
where we have used that $R$ commutes with $\sigma_0$. This commutation property is central for our separation of "Goldstone modes" and "massive modes" as done in eqs. (5.40) to (5.42) below; it is the motivation for the use of the decomposition (5.28). Combining eqs. (5.28) and (5.39), we have

$$\sigma = \sigma_G + \delta\sigma,$$

where

$$\sigma_G = T_0^{-1} \sigma_0^0 T_0.$$  \hspace{1cm} (5.41)

The index $G$ is introduced because the matrix $\sigma_G$ is analogous to the "Goldstone modes" of Schäfer and Wegner [2].

The form (5.37) is invariant under the transformation $\sigma \rightarrow T^{-1}\sigma T$. Therefore, with $\sigma_0^0$ all matrices $T^{-1}\sigma_0^0 T$ are saddle points of (5.37), too. These are precisely the matrices $\sigma_G$. This shows that the saddle-point manifold is parametrized in terms of the variables $(t_{12})$. The topological properties of this manifold have been discussed earlier in a simplistic fashion, and are taken up again in section 8.

To give a geometrical interpretation to the saddle-point manifold, we argue that the variables in $t_{12}$ define the saddle points, while the variables in $\delta P$ with

$$\delta \sigma = T_0^{-1} \delta P T_0$$  \hspace{1cm} (5.42)

define the directions of "steepest descent". (This is shown explicitly in section 6.) In other words: If we change the contours of integration in the integral (5.31) in such a way that we pass over a saddle point, then keeping the variables in $t_{12}$ fixed defines a fixed saddle point, while the variables in $P_{11}, P_{22}$ explore the vicinity of this saddle point in the direction of steepest descent. Changing $t_{12}$ we change the saddle point. All saddle points accessible by deformation of the original contours of integration are reached as the variables in $t_{12}$ range over the defining non-compact manifold.

In writing eq. (5.38), we have passed over an ambiguity in sign which we have commented on below eq. (5.26). While the sign factor multiplying $\Delta_0$ in eq. (5.38) is uniquely determined for the Boson–Boson part of $\sigma_0^0$ (cf. section 5.3), this is not so for the Fermion–Fermion part. We encounter here an ambiguity similar to the one displayed in section 5.4 and appendix E. Our choice (5.38) makes the projection of $\sigma_0^0$ onto the (1, 1) block and onto the (2, 2) block each a multiple of the unit matrix. This choice is tailored to the parametrization (5.28) and the factoring of the matrix $R$, see eq. (5.41). For a complete evaluation of the saddle-point integral it is necessary, however, to include all saddle points that can be reached by continuous deformation of the original path of integration. For the Fermion–Fermion block, this means that besides the saddle point with $\text{Re} \tau_1 < 0$ and $\text{Re} \tau_2 > 0$ contained in the parametrization (5.38), (5.41), we must also include the saddle points with $\text{Re} \tau_1 > 0$, $\text{Re} \tau_2 < 0$, with $\text{Re} \tau_1 > 0$ and $\text{Re} \tau_2 > 0$, and with $\text{Re} \tau_1 < 0$ and $\text{Re} \tau_2 < 0$. The first of these, $\text{Re} \tau_1 > 0$ and $\text{Re} \tau_2 < 0$, is actually part of the integration manifold $\sigma_G$. In the parametrization introduced in appendix I, it is reached when $u_1 = 1 = u_2$, $U = 1$ and $\mu = -1$. Therefore, this point need not be considered separately. The other two points, however, are not contained in $\sigma_G$. We show at the end of appendix F that these two saddle points yield a contribution which vanishes in the limit $N \rightarrow \infty$. Anticipating this result we do not consider them here.

Our parametrization of $\sigma$ near the saddle point in eqs. (5.40) to (5.42) is different from that of Schäfer and Wegner [2] but the same as the one used by Pruisken and Schäfer [16]. As shown in section
6 and appendix F, this parametrization makes the integration over $\delta P_{11}$ and $\delta P_{22}$ in the limit $N \to \infty$ trivial, as opposed to the integration over “massive modes” in the parametrization of Schäfer and Wegner. Moreover, the parametrization (5.30) for $\sigma$ leads naturally to the appearance in the volume element $d\mu(t)$ of the invariant measure associated with the transformation $T_0$, while in the parametrization of Schäfer and Wegner this measure appears as the result of the integration over massive modes.

6. The limit $N \to \infty$

In the limit $N \to \infty$, the integrals over the $P$-variables can be calculated with the help of the saddle-point approximation. This procedure is exact and does not depend on details of the effective Lagrangian such as sources and coupling to the channels. The limit $N \to \infty$ is taken keeping the number of channels fixed.

The effective Lagrangian in eq. (4.12) contains the term $(N/4) \text{tr} \sigma^2$. With $\sigma$ given by eqs. (5.40) to (5.42), we have $(N/4) \text{tr} \sigma^2 \to (N/4) \text{tr}[\delta P_{11}^2 + \delta P_{22}^2]$. [We have used that $\text{tr} (\sigma_0^2)^2 = 0$, and that linear terms in $\delta P_{pp}$ cancel against those originating from the ln-terms in eq. (4.12) by virtue of the saddle-point condition.] This suggests introducing $\delta P' = N^{1/2} \delta P$ as a new set of variables, expanding in powers of $\delta P'$ and omitting all terms which vanish in the limit $N \to \infty$. To carry out these steps, we must clarify which powers of $N$ are carried by the terms on the r.h.s. of eq. (4.10). We take $\lambda$ as independent of $N$; the same holds true of $E$ since $|E| \leq 2\lambda$ [we recall the definitions (3.12a)]. Since $\varepsilon$ is destined to probe fluctuation properties which are of the order of the mean level spacing $d \propto \lambda/N$, we take $\varepsilon$ to be of the order $N^{-1}$. Guided by the results of ref. [10], we expect the quantities $u_{\varepsilon}$ of eq. (2.7) to enter the final result in the form $u_{\varepsilon}^2/d \propto N u_{\varepsilon}^2 / \lambda$. Hence, $N u_{\varepsilon}^2$ is of order unity.

In eq. (4.12), we make the shift of variables

$$\sigma \to \sigma - \frac{1}{2\lambda} \varepsilon L. \quad (6.1)$$

This removes the $\varepsilon$-dependence from the logarithm. Recalling that $\varepsilon \propto N^{-1}$, dropping terms of order $N^{-1}$, and taking the limit $\eta \to 0$, we find for the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{N}{4} \text{tr} (\sigma^2) + \frac{N\varepsilon}{4\lambda} \text{tr} (\sigma L) - \frac{1}{2} \text{tr} \ln(E - \lambda \Sigma + iW + J). \quad (6.2)$$

We now write

$$\sigma = \sigma_0 + \delta \sigma = \sigma_0 + N^{-1/2} \delta \sigma'. \quad (6.3)$$

Introducing the decomposition (6.3) into eq. (6.2) we expand in powers of $\delta \sigma'$, keeping only terms which do not vanish in the limit $N \to \infty$. Since terms linear in $\delta \sigma'$ cancel by virtue of the saddle-point condition, this effectively amounts to keeping only terms of order zero and two in $\delta \sigma'$. Moreover, a term proportional to $N$ arises from the $\mu$-summation over the logarithmic expression in eq. (6.2) only in zeroth order in both $W$ and $J$. This yields
\[ \mathcal{L}_{\text{eff}} = -\frac{N}{4} \operatorname{trg}(\delta \sigma^2) + \frac{N}{4\lambda} \operatorname{trg}(\sigma_G L) - \frac{1}{2} \operatorname{trg} \ln(E - \lambda \Sigma_G + iW + J) + \frac{N\lambda^2}{4} \operatorname{trg} \left( (E - \lambda \sigma_G)^{-1} \delta \sigma \right)^2. \] (6.4)

Here \( \Sigma_G \) is given by \( \{\delta_{\mu\nu}(\sigma_G)_{\alpha\beta}\} \). We use the saddle-point condition,

\[ \sigma_G(E - \lambda \sigma_G) = \lambda. \] (6.5)

We also use eqs. (5.41) and (5.42) and the fact that \( [\sigma_D^0, \delta P] = 0 \). All this and eq. (5.38) yield

\[ \mathcal{L}_{\text{eff}} = -\frac{N}{2} \operatorname{trg} \left( \delta P \right)^2 \left( 1 - \frac{E^2}{4\lambda^2} + i \frac{E}{2\lambda} \Delta_{0L} \right) \right] + \mathcal{L}_{\text{eff}}^0(\sigma_G, J). \] (6.6)

The variables \( \sigma_G \) and \( \delta P \) are decoupled. This is true only in the Gaussian approximation, or for \( N \to \infty \). The effective Lagrangian \( \mathcal{L}_{\text{eff}}^0 \) depends only on the Goldstone modes \( \sigma_G \) and is given by

\[ \mathcal{L}_{\text{eff}}^0(\sigma_G, J) = \left( \frac{Ne}{4\lambda} \right) \operatorname{trg}(\sigma_G L) - \frac{1}{2} \operatorname{trg} \ln(E - \lambda \Sigma_G + iW + J). \] (6.7)

We observe that the variables \( \{\delta P\} \) occur quadratically in \( \mathcal{L}_{\text{eff}} \) with a weight factor which inside the GOE spectrum has a negative definite real part. Following eqs. (5.32) and (5.29), we take \( \{\delta P\} \) and \( \{t_{12}\} \) as the (independent) variables of integration. As shown in appendix F, the Jacobian \( \mathcal{F}(P) \) in eq. (5.35) is unity at and near the saddle-point manifold, except for terms which vanish as \( N \to \infty \). The symmetries of the transformation matrices \( R_1 \) and \( R_2 \) (as indicated in tables D.1 and E.1 for the corresponding generators) show that the matrices \( \delta P_{11} \) and \( \delta P_{22} \) (the latter except for phase factors multiplying the anticommuting variables) have the same symmetries as the corresponding (1, 1) and (2, 2) blocks of the matrix \( \sigma \) in table 4.3. As a result of all this, the integrals over \( \delta P_{11} \) and \( \delta P_{22} \) are straightforward Gaussian integrals and yield a factor four. This leaves us with the integration over \( d\mu(t) \). The integrand contains the convergence-generating term of eqs. (5.31), (5.33) evaluated at the saddle point. This term is proportional to \( N \). The limit \( \alpha \to 0 \) must be taken prior to the limit \( N \to \infty \). Since \( N \) occurs only in the combination \( Na \), we write \( \tilde{\alpha} = 2Na\Delta_0 \) and simply consider the limit \( \tilde{\alpha} \to 0 \). We thus have

\[ \lim_{N \to \infty} \tilde{Z}(E(1), E(2), J) = \lim_{\tilde{\alpha} \to 0} \int d\mu(t) \exp\{+\mathcal{L}_{\text{eff}}^0(\sigma_G, J) - \tilde{\alpha} \operatorname{trg}(t_{12}t_{21}) \}. \] (6.8)

The constant \( c \) multiplying the integral in eq. (6.8) has been inserted because the integral (5.31) which is used in the Hubbard–Stratonovitch transformation has not yet been normalized to unity.

Equation (6.8) constitutes an important intermediate result, and a very considerable simplification of the original expressions, involving only integrations over 8 commuting and 8 anticommuting variables. The steps performed in the present section correspond to the integration over massive modes in the formulation by Schäfer and Wegner [2]. Equation (6.8) shows that we have mapped the problem of calculating \( S_{ab}S^{*}_{ca} \) onto a zero-dimensional non-linear \( \sigma \)-model, see eq. (6.5). It is useful to check that the form (6.8), when used to calculate \( S_{ab} \) and \( S_{ab}S^{*}_{ca} \), is consistent with unitarity. This check is carried through in appendix G by means of a Ward identity.

Technically the limit \( N \to \infty \) taken in this section corresponds to calculating the term of lowest order in an (asymptotic) expansion in powers of \( N^{-1} \). The dimension \( N \) of the GOE Hamiltonian matrix
appears also as a factor multiplying $\varepsilon$, and as the limiting value of the $\mu$-summation, in eq. (6.7). In these two expressions, we continue to consider $N$ as a large but finite number. Eventually, $N$ will appear only in the combination $N\nu^2_n$ [this term is absorbed into the coefficient $x_n$, see eq. (7.7)] and in the combination $N\Delta_0/\lambda$ [this term is absorbed into the mean level spacing $d$, see eq. (7.22)].

7. Introduction of transmission coefficients

In the replica formalism, it was possible to express the fluctuations of the $S$-matrix in terms of the transmission coefficients $T_a$ defined by

$$T_a = 1 - |S_{aa}|^2.$$  

(7.1)

In the present section we show that the same is possible in the framework of eq. (6.8). We must first work out the average $S$-matrix. This is possible by repeating the steps leading to eq. (4.12) for the one-point function. Equivalently, we may drop the integrations over $d\sigma_{12}$ and $d\sigma_{22}$ (in $[1, 2]$ block notation) in eq. (4.12), and take the integral at $\sigma_{12} = 0 = \sigma_{22}$. (This latter procedure may be viewed as an application of Wegner's integral theorem [17].) This yields for the generating function $\tilde{Z}_1$ of the one-point function the expression (including a proper normalization constant)

$$\tilde{Z}_1(E(1), J) = \frac{1}{2} \int d[\sigma] \exp\{\mathcal{L}_{\text{eff}}(J)\},$$  

(7.2)

with an effective Lagrangian given by

$$\mathcal{L}_{\text{eff}}(J) = -\frac{N}{4} \sum_{\alpha} (\sigma^2) - \frac{1}{2} \sum_{\alpha, \mu} \ln(E \cdot 1 - \lambda\Sigma + M(J))$$  

(7.3)

where $1$, $\Sigma$ and $M(J)$ are graded $4N \times 4N$ matrices. We find for the saddle point the value

$$\sigma^0 = E/2\lambda - i\Delta_0.$$  

(7.4)

Expanding $\mathcal{L}_{\text{eff}}(J)$ in the vicinity of $\sigma^0$ for $N \gg 1$, we obtain as in section 6 with $\sigma = \sigma^0 + \delta\sigma$

$$\mathcal{L}_{\text{eff}}(J) \equiv -\frac{N}{4} \sum_{\alpha} (1 - (\sigma^0)^2) \ln(E \cdot 1 + M(J)).$$  

(7.5)

As in section 6, the Gaussian integral over $\delta\sigma$ converges if $E$ lies within the GOE spectrum and away from its end points, so that

$$\lim_{N \to \infty} \tilde{Z}_1(E, J) = \exp\left\{ -\frac{1}{2} \sum_{\alpha, \mu} \ln(E \cdot 1 + M(J)) \right\}.$$  

(7.6)

A straightforward calculation yields
\[ S_{ab} = \delta_{ab} \frac{1 - ix_{a}\sigma_{0}}{1 + ix_{a}\sigma_{0}} \]  \text{with} \quad x_{a} = \pi N_{\beta}^{2} / \lambda. \] (7.7)

The average S-matrix is diagonal, in keeping with remarks in section 2. The transmission coefficient defined in eq. (7.1) is given by

\[ T_{\alpha} = \frac{4x_{\alpha}A_{0}}{1 + 2x_{\alpha}A_{0} + x_{\alpha}^{2}}, \] (7.8)

where \( A_{0} \) is proportional to the level density of the GOE, cf. eqs. (5.19) and (7.22).

The effective Lagrangian of eq. (6.7) contains one part (the logarithm) which depends on the coupling to the channels. Using the saddle-point eq. (6.5), we write this part in the form

\[ -\frac{1}{2} \text{trg} \ln(1 + i\lambda^{-1}\sigma_{G}W + \lambda^{-1}\sigma_{G}J). \] (7.9)

According to eq. (3.16c), the two-point function is obtained by double differentiation with respect to \( J \) of \( \tilde{Z}(E(1), E(2), J) \) at \( J = 0 \). We therefore expand the term (7.9) in powers of \( J \), insert the result into eq. (6.8) and expand the exponential in powers of \( J \). This yields

\[ \tilde{Z}(E(1), E(2), J) = \lim_{\tilde{\alpha} \to 0} c \int d\mu(t) \left\{ \frac{1}{4} \text{trg}[(\rho \lambda^{-1}\sigma_{G}J)^{2}] + \frac{1}{8} \text{trg}(\rho \lambda^{-1}\sigma_{G}J)^{2} + \cdots \right\} \exp \left[ \frac{N_{E}}{4\lambda} \text{trg}(\sigma_{G}L) \right. \]

\[ \left. -\frac{1}{2} \text{trg} \ln(1 + i\lambda^{-1}\sigma_{G}W) - \tilde{\alpha} \text{trg}(t_{12}t_{21}) \right]. \] (7.10)

The dots indicate terms which vanish as we apply eq. (3.16c). We have defined

\[ \rho = (1 + i\lambda^{-1}\sigma_{G}W)^{-1}. \] (7.11)

The result of the differentiation (3.16c) simplifies if we calculate \((S_{ab}(E(1)) - \delta_{ab})(S_{cd}^{*}(E(2)) - \delta_{cd})\). To write the result in compact form, we define the \( 8N \times 8N \) graded matrices \( R_{ab}(p) \) with \( a, b = 1, \ldots, A \) by

\[ R_{ab}(p) = \{ I_{ab}(p) (W_{\mu\lambda} W_{ab} + W_{ab} W_{\mu\lambda}) \}. \] (7.12)

Here, \( I_{ab}(p) \) is the projection of the graded \( 8 \times 8 \) matrix \( I \) defined below eq. (3.12c) onto the \( (p, p) \) block with \( p = 1 \) or \( p = 2 \). We have

\[ \overline{(S_{ab}(E(1)) - \delta_{ab})(S_{cd}^{*}(E(2)) - \delta_{cd})} = \lim_{\tilde{\alpha} \to 0} c \int d\mu(t) \left\{ \frac{\pi^{2}}{8} \text{trg}[\rho \lambda^{-1}\sigma_{G} R_{ab}(1) \rho \lambda^{-1}\sigma_{G} R_{cd}(2)] \right. \]

\[ + \frac{\pi^{2}}{16} \text{trg}[\rho \lambda^{-1}\sigma_{G} R_{ab}(1)] \text{trg}[\rho \lambda^{-1}\sigma_{G} R_{cd}(2)] \}

\[ \times \exp \left[ \frac{N_{E}}{4\lambda} \text{trg}(\sigma_{G}L) - \frac{1}{2} \text{trg} \ln(1 + i\lambda^{-1}\sigma_{G}W) - \tilde{\alpha} \text{trg}(t_{12}t_{21}) \right]. \] (7.13)
We refer to the two terms in curly brackets as the pre-exponential factors. To write these terms and the term \(- \frac{1}{2} \trg \ln(1 + i \lambda^{-1} \sigma_G W)\) in the exponent as functions of the transmission coefficients, we expand each term formally in powers of \(W\), use the definition (3.12b) for \(W\), and carry out the \(\mu\)-summations with the help of eq. (2.7), using the definition (7.7) for \(x_a\). The resulting series can be resummed. This yields

\[
- \frac{1}{2} \sum_{a=1}^{A} \trg \ln(1 + i x_a \sigma_G L),
\]

(7.14a)

\[
\trg (\rho \lambda^{-1} \sigma_G R_{ab}(1)) = \frac{2}{\pi} \delta_{ab} \trg[(1 + i x_a \sigma_G L)^{-1} x_a \sigma_G I(1)],
\]

(7.14b)

an analogous expression for the term involving \(R_{cd}(1)\), and

\[
\trg (\rho \lambda^{-1} \sigma_G R_{ab}(1) \rho \lambda^{-1} \sigma_G R_{cd}(1)) = \frac{1}{\pi^2} \left( \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} \right) \{ \trg[(1 + i x_a \sigma_G L)^{-1} x_a \sigma_G I(1)] \times (1 + i x_b \sigma_G L)^{-1} x_b \sigma_G I(2) \} + (a \leftrightarrow b).
\]

(7.14c)

Here and in the sequel we suppress the index below the symbol \(\trg\) as we henceforth take only traces of graded 8 \(\times\) 8 matrices. We write the graded 8 \(\times\) 8 matrix \((1 + i x_a \sigma_G L)\) as

\[
1 + i x_a \sigma_G L = \begin{pmatrix} e & f \\ g & h \end{pmatrix},
\]

(7.15)

where \(e, f, g, h\) are graded 4 \(\times\) 4 matrices, and where we have used [1, 2] block notation. Writing

\[
\trg \ln \begin{pmatrix} e \\ g \\ f \\ h \end{pmatrix} = \trg \ln \begin{pmatrix} e \\ \emptyset \\ h \\ f \end{pmatrix} + \trg \ln \left( \begin{pmatrix} 1 & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & e^{-1} f \end{pmatrix} \right),
\]

(7.16)

we expand the second logarithm on the r.h.s of eq. (7.16) in powers of \(e^{-1} f\) and \(h^{-1} g\). Since the graded trace of all odd powers vanishes, and since the sum over the even powers can be carried out, we find

\[
\trg \ln(1 + i x_a \sigma_G L) = \frac{1}{2} \trg \ln \begin{pmatrix} e^2 - efh^{-1}g & \emptyset \\ \emptyset & h^2 - hge^{-1}f \end{pmatrix}.
\]

(7.17)

Analogously,

\[
(1 + i x_a \sigma_G L)^{-1} = \begin{pmatrix} 1 & -e^{-1} f \\ -h^{-1} g & 1 \end{pmatrix} \begin{pmatrix} e - fh^{-1} g & \emptyset \\ \emptyset & h - ge^{-1} f \end{pmatrix}^{-1}.
\]

(7.18)

To evaluate the r.h.s. of eqs. (7.17) and (7.18), we use eq. (5.41) for \(\sigma_G\), eq. (5.29) for \(T_0\), the definition (7.15), and the fact that for any analytic function \(F\), we have
\[ t_{12} F(t_{21} t_{12}) = F(t_{12} t_{21}) t_{12} . \]

(7.19)

(This can be verified by a formal power-series expansion.) We also use the cyclic invariance of the graded trace, eqs. (7.7) and (7.8), the fact that \( \text{trg} I(1) = \text{trg} I(2) = -4 \) and the definitions

\[ \alpha_1 = 2t_{12} t_{21} ; \quad \alpha_2 = 2t_{21} t_{12} . \]

(7.20)

We find

\[ -\frac{1}{2} \sum_a \text{trg} \ln(1 + ix_a \sigma_a L) = -\frac{1}{2} \sum_a \text{trg} \ln(1 + \frac{1}{2} T_a \alpha_1), \]

(7.21a)

\[ \text{trg}[ (1 + ix_a \sigma_a L)^{-1} x_a \sigma_a I(1) ] = -2i(S_{aa} - 1) - \frac{i}{4} S_{aa} T_a \text{trg}[ (1 + \frac{1}{2} T_a \alpha_1)^{-1} \alpha_1 I(1) ], \]

(7.21b)

\[ \text{trg}[ (1 + ix_a \sigma_a L)^{-1} x_a \sigma_a I(2) ] = 2i(S_{cc}^* - 1) + \frac{i}{4} S_{cc}^* T_c \text{trg}[ (1 + \frac{1}{2} T_c \alpha_2)^{-1} \alpha_2 I(2) ], \]

(7.21c)

\[ \text{trg}[ (1 + ix_a \sigma_a L)^{-1} x_a \sigma_a I(1) (1 + ix_b \sigma_b L)^{-1} x_b \sigma_b I(2) ] \]

\[ = \frac{1}{2} T_a T_b \text{trg}[ t_{21} (1 + \frac{1}{2} \alpha_1)^{1/2} (1 + \frac{1}{2} T_a \alpha_1)^{-1} I(1) t_{12} (1 + \frac{1}{2} \alpha_2)^{1/2} (1 + \frac{1}{2} T_b \alpha_2)^{-1} I(2) ]. \]

(7.21d)

All graded traces on the r.h.s. of eqs. (7.21) are taken of graded 4 \times 4 matrices. We do not indicate this obvious fact by special notation. Collecting everything, and using the fact that the average level spacing \( d \) of the GOE is given by [1]

\[ d = \pi \lambda / N \Delta_0 , \]

(7.22)

we find

\[ (S_{ab}(E(1)) - \delta_{ab})(S_{cd}(E(2)) - \delta_{cd}) = \lim_{\alpha \to 0} c \int \mu(t) (\delta_{ab}(S_{aa} - 1) + \frac{i}{8} S_{aa} T_a \text{trg}[ (1 + \frac{1}{2} T_a \alpha_1)^{-1} \alpha_1 I(1)] \]

\[ \cdot \delta_{cd}(S_{cc}^* - 1) + \frac{i}{8} S_{cc}^* T_c \text{trg}[ (1 + \frac{1}{2} T_c \alpha_2)^{-1} \alpha_2 I(2)] \]

\[ + \frac{1}{32}(\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) T_a T_b \text{trg}[ t_{21} (1 + \frac{1}{2} \alpha_1)^{1/2} (1 + \frac{1}{2} T_a \alpha_1)^{-1} I(1) \]

\[ \cdot t_{12} (1 + \frac{1}{2} \alpha_2)^{1/2} (1 + \frac{1}{2} T_b \alpha_2)^{-1} I(2)] + (a \leftrightarrow b)) \]

\[ \cdot \exp \left\{ -\frac{i \pi e}{2d} \text{trg} \alpha_1 - \frac{1}{2} \sum_a \text{trg} \ln(1 + \frac{1}{2} T_a \alpha_1) - \frac{1}{2} \alpha \text{trg} \alpha_1 \right\} . \]

(7.23)

Equation (7.23) bears a close formal analogy to eq. (6.10) of ref. [17]. It has two noteworthy features. First, the r.h.s. of eq. (7.23) does not depend explicitly on the energy \( E \) defined in eq. (3.12a), and is a function only of \( S_{aa} \), \( a = 1, \ldots, \Lambda \), and of the mean level spacing \( d \). In units \((\epsilon / d)\), the two-point function is the same throughout the GOE spectrum. This result—stationarity of the second moment of
\(S_{ab}(E)\) is analogous to a corresponding statement for GOE eigenvalue fluctuations \([1]\). It goes beyond the result of ref. \([10]\) which was restricted to the point \(E = 0\). Second, aside from trivial factors \(S_{aa}\) and \(S_{bc}^*\), the r.h.s. of eq. (7.23) depends only on the transmission coefficients \(T_a\), \(a = 1, \ldots, A\), and not on the phases of the individual average \(S\)-matrix elements. This feature is a consequence of the orthogonal symmetry of the GOE \([13]\).

Equation (7.23) has been derived in the framework of approximation (2.5). In appendix H we show that this approximation can be removed, and that the real parts of \(F_{\mu\nu}\) (the “shift functions”) can be included in the derivation. This leads to a different relation connecting the coefficients \(x_a\) with \(S_{aa}\) (eq. (7.7)). The connection (7.8) between the transmission coefficients and the coefficients \(x_a\) is likewise modified. However, the formula (7.23) expressing the two-point function in terms of the average \(S\)-matrix elements and the transmission coefficients remains unchanged. This shows that eq. (7.23) is of general validity.

As in ref. \([10]\), there are two contributions to the two-point function. One, proportional to \((\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\), is found to give the only surviving contribution in the limit \(\Lambda \gg 1\): It yields the Hauser—Feshbach formula as the leading-order term in an asymptotic expansion in powers of \((\Sigma, T_e)^{-1}\). The other one contributes only to elastic scattering and increases the elastic enhancement factor above the value of two which it attains in the limit \(\Lambda \gg 1\).

8. Integration over the saddle-point manifold

In eq. (7.23), the two-point function is given as an integral over 8 commuting and 8 anticommuting variables. In this section, we carry out the 8 integrals over Grassmann variables and 5 of the 8 integrals over commuting variables, thereby reducing the two-point function to a threefold integral over ordinary (commuting) variables. The resulting formula, eq. (8.10), was published in ref. \([18]\). This reduction is achieved by following the method of Efetov \([11]\) who suggested to introduce as independent variables of integration the “eigenvalues” of the matrix \(t_{12}\), and the angles of the associated transformation matrices. All commuting variables introduced in this way still contain an even polynomial in the Grassmann variables occurring in the original parametrization of \(t_{12}\) as given in table D.3. It turns out that the terms of zeroth order in the Grassmann variables play a special role. We adopt the standard nomenclature and refer to such terms for brevity as the “ordinary part” of the associated variable, abbreviating it by the symbol ord.

The “diagonalisation” of \(t_{12}\) (and of \(t_{21}\)) is carried out in appendix I. Equations (1.18) show that

\[
t_{12} = u_1^{-1} U \mu_0 u_2; \quad t_{21} = u_2^{-1} \mu_0 U^\dagger u_1.
\]  

(8.1)

The matrix \(\mu_0\) is diagonal and contains as elements the “eigenvalues” \((\mu_1, \mu_2, i\mu_1, i\mu_2)\) of both \(t_{12}\) and \(t_{21}\). The matrix \(U\) is an element of the group SU(2). The matrices \(u_1\) and \(u_2\) are each the product of a matrix \(O_p\) \((p = 1, 2)\) (an element of the group of orthogonal transformations in 2 dimensions), and of a matrix \(v_p\) \((p = 1, 2)\) which carries the anticommuting variables, see eq. (1.13). The matrices \(v_p\) are defined in eqs. (K.26), (K.25) and (K.23). Since \(U\) commutes with \(\mu_0\), eqs. (8.1) and the definitions (7.20) imply

\[
\alpha_1 = 2u_1^{-1} \mu_0^{-1} u_1; \quad \alpha_2 = 2u_2^{-1} \mu_0^{-1} u_2.
\]  

(8.2)

It is shown in appendix I that the transformation to diagonal form, eqs. (8.1) and (8.2), is possible unless
ord(μ₁) = 0 = ord(μ), or ord(μ₂) = 0 = ord(μ), or both, see eqs. (I.20). We return below to these singular points of the transformations (8.1), (8.2).

We use eqs. (8.1) and (8.2) in eq. (7.23). The terms in the exponent contain only graded traces of functions of α. The cyclic invariance of the graded trace and eq. (8.2) show that everywhere in the exponent, we may replace α by 2μ₂. Because of the occurrence of the matrices I(μ), (p = 1, 2) this statement does not apply to the pre-exponential factors. Inspection of these factors and of eqs. (8.1) and (8.2) shows that the matrices uₚ occur only in the combination uₚ I(μ) u⁻¹ₚ = Oₚ uₚ I(μ) v⁻¹ₚ Oₚ. We use the expansions (K.26), the identity Yₚ I(μ) = −I(μ) Yₚ (cf. eq. (K.25)), and the fact (demonstrated in appendix L) that only the terms of fourth order in Yₚ give non-vanishing contributions to eq. (7.23). Using eqs. (K.25) and (K.23), we find that these fourth-order terms are given by 2I(μ) Yₚ = 4μₚ αₚ βₚ βₚ.

These terms are multiples of the unit matrix, they commute with Oₚ and with U, and we arrive at the substitution rules

\[
\begin{align*}
&u_1 I(1) u_1^{-1} \rightarrow 4\hat{\alpha}_1 \hat{\beta}_1, \\
&U u_2 I(2) u_2^{-1} U^\dagger \rightarrow 4\hat{\alpha}_2 \hat{\alpha}_2 \hat{\beta}_2 \hat{\beta}_2, \\
&u_2 I(2) u_2^{-1} \rightarrow 4\hat{\alpha}_2 \hat{\beta}_2 \hat{\beta}_2.
\end{align*}
\]

The volume element dμ(t) is calculated in appendix K. According to eqs. (K.34) and (K.30), it is given by

\[
dμ(t) = \frac{(1 - μ^2)}{(1 + μ_1^2)^{1/2} (1 + μ_2^2)^{1/2}} \cdot \frac{μ^3 |μ^2 - μ_2^2|}{(μ_1^2 + μ_2^2)^2 (μ_2^2 + μ^2)^3 [1 + m^2 + r^2 + s^2]^{-2}} \\
\cdot dφ₁ dφ₂ dm dr ds dμ₁ dμ₂ dα₁ dα₁ dα₂ dα₂ dβ₁ dβ₂ dβ₁ dβ₂.
\]

The range of the ordinary parts of the commuting integration variables is as follows:

\[
0 \leq \text{ord}(φ_p) \leq 2π, \quad p = 1, 2; \\
−∞ < \text{ord}(m), \text{ord}(r), \text{ord}(s) < +∞; \\
0 \leq \text{ord}(μ) \leq 1; \quad −∞ < \text{ord}(μ₁), \text{ord}(μ₂) < +∞.
\]

These relations are implied by the derivation in appendices I and K. The relations (8.5) reflect the structure of the saddle-point manifold. We emphasize especially the infinite range of μ₁, μ₂ and the finite range of μ. This difference in range mirrors the non-compactness of the saddle-point manifold with respect to the Boson–Boson block, and its compactness with respect to the Fermion–Fermion block. The angle variables φₚ parametrize the real orthogonal matrices Oₚ, and the real variables m, r and s parametrize the SU(2) matrix U.

We show in appendix L that

\[
\int dμ(t) \exp\left\{-\frac{iπε}{2d} \text{trg} α_1 - \frac{1}{2} \sum a \text{trg} \ln(1 + \frac{1}{2} T_a a_1) - \frac{1}{2} \hat{α} \text{trg} a_1\right\} = 1.
\]

Equation (8.6a) holds for all values of ε, and of the Tₐ. The proof of eq. (8.6a) is based on the invariance
of the integrand under graded unitary transformations of $t_{12}$. Equation (8.6a) also implies that the normalization constant $c$ in eq. (7.23) has the value unity,

$$c = 1.$$  \hspace{1cm} (8.6b)

Indeed, $c$ must be chosen in such a way that $\bar{Z}(E(1), E(2), 0) = 1$. Comparing eq. (6.8) with eq. (8.6a) we see that the integral in eq. (6.8) is unity as it stands, so that eq. (8.6b) follows.

It is also easy to see that for all channels $a$, for all values of $S_{aa}$ and $T_a$, and for $p = 1$ and 2, we have

$$\overline{S}_{aa} T_a \int d \mu(t) \text{trg} \left[ (1 + \frac{1}{2} \alpha_p T_a)^{-1} \alpha_p I(p) \right] \exp(\ldots) = 0.$$  \hspace{1cm} (8.7)

The dots in the exponent indicate the same terms as occur in the exponent of eq. (8.6a). It is shown in appendix L that eq. (8.7) can be derived in very much the same way as eq. (8.6a). An indirect proof consists in repeating the steps which lead from eq. (6.8) to eq. (7.23) for the one-point function $(S_{ab} - \delta_{ab})$. The result has the form of the r.h.s. of eq. (7.23), with the pre-exponential factors (the terms in curly brackets) replaced by

$$\delta_{ab} \left( (S_{aa} - 1) + \frac{1}{2} S_{aa} T_a \text{trg} \left[ (1 + \frac{1}{2} T_a \alpha_1)^{-1} \alpha_1 I(1) \right] \right),$$

and analogously for $(S^*_{cd} - \delta_{cd})$. Equations (8.6) then imply eq. (8.7).

We now use eqs. (8.3) to (8.7) to simplify the r.h.s. of eq. (7.23). As a result, we find that the r.h.s. consists of two terms. The first one is given by $(S_{aa} - 1) (S^*_{cd} - 1) \delta_{ab} \delta_{cd}$. It yields the disconnected contribution to the two-point function. By virtue of the substitutions (8.3), the integrand of the second term contains the factor $\prod_{p=1}^{\infty} \frac{\bar{\alpha}_p \alpha_p}{\bar{\beta}_p \beta_p}$. This fact simplifies the calculation enormously: All commuting integration variables can be replaced by their ordinary parts. The integrations over the 8 anticommuting variables can then be carried out immediately and yield a factor of $(2\pi)^{-4}$. To simplify the notation, we henceforth omit the symbol "ord" since Grassmann variables no longer appear in the formalism. We obtain

$$\overline{S}_{ab} (E(1)) S^*_{cd} (E(2)) - \overline{S}_{ab} \overline{S}^*_{cd}$$

$$= 4 (2\pi)^{-4} \lim_{\alpha \to 0} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \int_0^{2\pi} d\varphi_3 \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \int_{-\infty}^{\infty} d\mu_3$$

$$\cdot \exp \left\{ -\frac{i\pi \varepsilon}{d} \text{trg} \mu_3^2 + \frac{1}{2} \sum_c \text{trg} \ln(1 + T_c \mu_3^2) - \bar{\alpha} \text{trg} \mu_3^2 \right\}$$

$$\cdot \{ \delta_{ab} \delta_{cd} \overline{S}_{aa} S^*_{cc} T_a T_c \text{trg} \left[ (1 + T_a \mu_3^2)^{-1} \mu_3^2 \right] \text{trg} \left[ (1 + T_c \mu_3^2)^{-1} \mu_3^2 \right]$$

$$+ (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) T_a T_b \text{trg} \left[ \mu_3^2 (1 + \mu_3^2) (1 + T_a \mu_3^2)^{-1} (1 + T_b \mu_3^2)^{-1} \right] \}.$$  \hspace{1cm} (8.8)
The integrals over $\varphi_1$, $\varphi_2$, $m$, $r$ and $s$ can be carried out trivially and yield the factor $(2\pi)^2 4\pi (\pi/4) = 4\pi^4$.

We recall that the transformation to diagonal form, eqs. (8.1) and (8.2), is singular for $\mu_1 = \mu = 0$, for $\mu_2 = 0$, and for $\mu_1 = \mu_2 = 0$, see eqs. (1.20). It is easy to see that this singularity does not correspond to a singularity of the integrand in eq. (8.8), so that this latter equation is justified as it stands. [For $\mu_1 = 0 = \mu \neq \mu_2$, we put in the vicinity of zero $\mu_1 = t \cos \varphi$, $\mu = t \sin \varphi$. The overall volume element then carries the power $(t)^0 = 1$ and is therefore non-singular. For $\mu_1 = 0 = \mu_2 = \mu$, a similar parametrization yields for the volume element the factor $t^{-1}$. This factor, however, is overcompensated by a factor $t^2$ arising from the last curly bracket in eq. (8.8).]

We introduce new integration variables,

$$\lambda_p = \mu_p^2, \quad p = 1, 2; \quad \lambda = \mu^2.$$  

(8.9)

Equation (8.8) takes the form

$$\overline{S_{ab}}(E(1)) S_{cd}^*(E(2)) - \overline{S_{ab}} S_{cd}^* = \frac{1}{8} \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \int_0^1 d\lambda \frac{(1 - \lambda)\lambda}{(1 + \lambda_1)\lambda_1^2} \frac{(1 - \lambda_2)\lambda_2}{(1 + \lambda_2)\lambda_2^2}$$

$$\cdot \exp \left\{ -\frac{i\pi}{d} (\lambda_1 + \lambda_2 + 2\lambda) \right\} \Pi_e \frac{(1 - T_\lambda)}{(1 + T_\lambda)^{1/2} (1 + T_\lambda)^{1/2}}$$

$$\cdot \left\{ \delta_{ab}\delta_{cd} \overline{S_{aa}} S_{cc}^* T_a T_c \left( \frac{\lambda_1}{1 + T_\lambda \lambda_1} + \frac{\lambda_2}{1 + T_\lambda \lambda_2} + \frac{2\lambda}{1 - T_\lambda} \right) + (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) T_a T_b \right\}.$$  

(8.10)

For $\lambda_p \to \infty$, the integral is convergent as it stands. This is why we have taken the limit $\lambda \to 0$ before carrying out the integration. Equation (8.10) constitutes the final result of our paper. It coincides with eq. (3) of ref. [18], save for two misprints in this formula (incorrect indices for the transmission coefficients in the source term).

9. Summary and conclusions

The result of this paper is embodied in eq. (8.10). In the derivation of this equation we have, however, produced a number of additional results which have relevance beyond the derivation of a closed expression for average compound-nucleus scattering cross sections.

The derivation of eq. (8.10) involves five major steps. For a suitably defined stochastic model (section 2) of the scattering matrix based on a GOE Hamiltonian, we have introduced a generating function
involving both commuting and anticommuting integration variables (section 3). The calculation of the ensemble average—a trivial manipulation in the present context—was followed by the Hubbard-Stratonovitch transformation. This is a major step with implications beyond the nuclear reaction problem: Extending the arguments of Schäfer and Wegner to the case of anticommuting variables, and using the symmetry of the Gaussian Orthogonal Ensemble and the convergence of the integrals as guiding principles, we have shown that the parametrization of the composite variables $\sigma$ is given in terms of a group of graded matrices. This group is compact with respect to the Fermion–Fermion block, and non-compact with respect to the Boson–Boson block. This result is valid in general and for a wide class of problems involving GOE Hamiltonians. The proof in sections 4, 5 and appendices D and E constitutes a major part of this paper. The third step, carried out in section 6 and appendix F, consists in the integration over massive modes. It is based on a saddle-point approximation and the limit $N \to \infty$, and on a decomposition of integration variables which heavily uses the symmetries underlying the parametrization of $\sigma$. This step leads to the second important and general result of the paper—a mapping of the original stochastic problem onto a non-linear $\sigma$-model as given by eqs. (6.5) and (6.8). This $\sigma$-model involves both commuting and anticommuting variables. It has dimension zero. This is the reason why it can be solved exactly, and why we eventually obtain the closed expression (8.10). The connection between the non-linear $\sigma$-model and the original stochastic problem is again quite general and not restricted to the nuclear scattering problem. In step four, carried out in section 7 and appendix H, we express the two-point function (aside from trivial $S$-matrix factors) in terms of the transmission coefficients. The significance of this result, which is obviously of interest only to the nuclear scattering problem, is twofold. First, it shows that average compound–nucleus scattering cross sections depend only on the transmission coefficients, not on the phases of the individual average $S$-matrix elements. Second, the result shows that the fluctuation properties of the $S$-matrix—when expressed in terms of the transmission coefficients—are the same over the entire GOE spectrum. This stationarity, while not unexpected, enhances our belief that the GOE provides a realistic model of actual cross-section fluctuations. The execution of step four again heavily uses the symmetries of the parametrization of $\sigma$, here especially those of the saddle-point manifold. The fifth and last step consists in carrying out the remaining integrals over the saddle-point manifold inasmuch as this seemed possible analytically. The calculation is lengthy and the technicalities, contained in section 8 and appendices I, K and L, are non-trivial. This step, too, can only be taken if one utilizes the symmetries of the problem to their full extent. Most steps in the calculation of the integrals are again generally useful for a wide class of GOE problems and beyond.

As a future realm of application, we mention the theory of localization in extended disordered systems. By analogy with the work of Schäfer and Wegner, the analysis of sections 4, 5, 6 and 8 and appendices D, F, I, K and L is fully applicable to Wegner's $N$-orbital model [19] in the limit $N \to \infty$. In the vicinity of the mobility edge, it also applies to the case $N = 1$. The results obtained in appendix L offer a new method for developing a systematic expansion ("strong-coupling expansion") around the localized regime. Work in this direction is under way.

From the point of view of stochastic physics, eq. (8.10) constitutes the answer to a well-defined problem which turns out to have a solution in closed form. The interest is focussed as much on the results obtained along the way—the parametrization of $\sigma$, the structure of the saddle-point manifold, and the evaluation of the various integrals—as on the result itself. From the point of view of compound–nucleus scattering theory, eq. (8.10) requires further analysis. What is the connection between the present approach and the maximum-entropy formulation of the same problem given by the Mexican group [7]? What can we learn from eq. (8.10) about the dependence of elastic enhancement
factors on the number of open channels, and on the values of the transmission coefficients? (This question is important for applications.) How does the result (8.10) compare with numerical Monte-Carlo evaluations of the stochastic model (2.2)? The answers to these and related questions (which we are now in the process of studying) exceed the frame of even this paper and will be communicated elsewhere.

We believe the present work to have useful applications in three areas. As explained above, our results are hopefully of general interest to theorists working on problems of disordered solids. Inasmuch as the GOE Hamiltonian encapsulates properties of the many-body problem which are the quantum analogues of non-integrability in classical systems, we are confident that our work will be relevant to the many-body theory of microscopically small systems. We have in mind, for instance, the theory of the effective interaction. Finally, it is obvious that an extension of the method would be applicable to other problems in statistical nuclear reaction theory. We mention only the four-point function, the breaking of time-reversal invariance, and isospin mixing in compound-nucleus reactions, as examples.

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Appendix A. Graded vectors and graded matrices: Conventions and definitions

The paper by Efetov [11] contains a useful introduction to the usage of anticommuting variables, including vectors and matrices some elements of which are anticommuting. The reader who is not familiar at all with this method is advised to study Efetov’s introduction. Unfortunately, the conventions introduced by Efetov differ in the relative ordering of commuting and anticommuting variables and, as a consequence, in the signs of several operations, from the standard conventions used by many authors of papers on field theory and statistical mechanics. In order to remain in the mainstream of the current literature on the subject, we have opted for these latter, standard conventions which to us appear also to be the more natural ones. Unfortunately, we are not aware of a complete introductory paper on the subject using these latter conventions. This is why we refer the reader to Efetov’s paper [11]. Since our use of a convention different from ref. [11] may result in confusion, we collect here the most relevant definitions which we use.

We consider \( N \) anticommuting variables \( \chi_i, i = 1, \ldots, N \) with

\[
\chi_i \chi_j = -\chi_j \chi_i, \quad 1 \leq i < j \leq N \tag{A.1}
\]

and

\[
\chi_i^2 = 0, \quad i = 1, \ldots, N. \tag{A.2}
\]

The adjoint (or "complex conjugate") of a variable \( \chi_i \) is written as \( \chi_i^* \). The variable \( \chi_i^* \) is independent of \( \chi_i \) for all \( i \) so that \( \chi_i^* \chi_j = -\chi_j \chi_i^* \neq 0 \). The adjoint operation \( ^* \), \( \chi_i \to \chi_i^* \), can be extended to the entire
Grassmann algebra in two different ways. We find it convenient to use the adjoint of the second kind which is defined by [20]

\[(\chi)\ast = \chi\ast, \quad (A.3)\]

\[(\chi^\ast)\ast = -\chi, \quad (A.4)\]

\[(\chi\chi^\ast)\ast = \chi^\ast\chi\ast. \quad (A.5)\]

This set of conventions is to be contrasted with the adjoint of the first kind where the minus sign in eq. (A.4) is omitted, and the ordering of the Grassmann variables reversed on the r.h.s. of eq. (A.5).

We also work with ordinary commuting variables. In this appendix, only real commuting variables are considered. The generalization to complex variables is straightforward but not needed in the present context. A graded vector \(\phi\) is an array of commuting \((S_i)\) and anticommuting \((\chi, \chi^\ast)\) variables,

\[
\phi = \begin{bmatrix}
S_1 \\
\vdots \\
S_{2L} \\
\chi_1 \\
\vdots \\
\chi_L \\
\chi_1\ast \\
\vdots \\
\chi_L\ast
\end{bmatrix}.
\quad (A.6)
\]

For simplicity we consider only vectors with an equal number \(2L\) of commuting and anticommuting elements. The number \(L\) is arbitrary. It turns out that \(L = N\) for the one-point function, and \(L = 2N\) for the two-point function, with \(N\) the dimension of the GOE matrix \(H_{ij}\). Some of the definitions introduced below depend on the relative order of commuting and anticommuting variables in \(\phi\). Care has to be taken in the formal manipulations regarding signs whenever the sequence of arguments in expression (A.6) is reordered.

The transpose of a graded vector, denoted by the superscript T, is defined as for ordinary vectors,

\[
\phi^T = (S_1, \ldots, S_{2L}, \chi_1, \ldots, \chi_L, \chi_1\ast, \ldots, \chi_L\ast).
\quad (A.7)
\]

The Hermitian adjoint, denoted by a dagger, is obtained by combining transposition and complex conjugation:

\[
\phi^\dagger = (\phi^T)^\ast = (S_1, \ldots, S_{2L}, \chi_1\ast, \ldots, \chi_L\ast, -\chi_1, \ldots, -\chi_L).
\quad (A.8)
\]

The definition (A.8) yields for the scalar product
(\phi^\dagger \cdot \phi) = \sum_{\mu=1}^{2L} S_{a_{\mu}}^2 + 2 \sum_{\mu=1}^{L} \chi_{\mu}^* \chi_{\mu}.

(A.9)

This expression does not change under complex conjugation.

Let F denote a matrix of dimension 4L. Defining the product F\phi in the usual way, and requiring that F\phi be again a graded vector, we are led to write F in the form

\[ F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \]

where \(a, b, \sigma, \rho\) are 2L \times 2L matrices, where \(a\) and \(b\) have as elements commuting variables, while \(\rho\) and \(\sigma\) have as elements anticommuting variables. (Note that the product of two anticommuting variables is a commuting variable!) Such matrices are called graded matrices ("supermatrices").

Following Efetov [11], we define the transpose \(F^T\) of a graded matrix \(F\) by the condition that

\[ \phi^T F^T = (F\phi)^T \]

(A.11)

for any graded vector \(\phi\). This implies

\[ F^T = \begin{pmatrix} a^T & \rho^T \\ -\sigma^T & b^T \end{pmatrix}. \]

(A.12)

Here, \(a^T, b^T, \rho^T, \sigma^T\) are transposed matrices constructed according to the standard rules. Note the minus sign in eq. (A.12) which makes \(F^T\) differ from the transpose of an ordinary matrix. The definition (A.12) implies that the Hermitean adjoint of a graded matrix \(F\), denoted by a dagger and defined by

\[ F^\dagger = (F^T)^*, \]

(A.13)

is explicitly given by

\[ F^\dagger = \begin{pmatrix} a^\dagger & \rho^\dagger \\ -\sigma^\dagger & b^\dagger \end{pmatrix} \]

(A.14)

if \(F\) has the form (A.10). Again, \(a^\dagger, b^\dagger, \rho^\dagger, \sigma^\dagger\) are Hermitean adjoints of \(a, b, \rho, \sigma\) defined as usual.

A graded matrix \(H\) with \(H^\dagger = H\) is called Hermitian. The definitions just given imply that for any graded vector \(\phi\), the matrix \(F\) defined as the dyadic product \(\phi F\) is Hermitean.

We note that the elements of the matrices \(a, b\) need not consist of ordinary real variables only. Given an algebra of anticommuting variables, these matrices may also contain polynomials of even order in the anticommuting variables with real coefficients. Likewise, the matrices \(\rho\) and \(\sigma\) have elements which consist of polynomials of odd order in the anticommuting variables. The complex conjugation involved in taking the Hermitean adjoints \(a^\dagger, b^\dagger, \rho^\dagger, \sigma^\dagger\) must obey the conventions (A.4) and (A.5).

Unitary graded matrices are defined in analogy to the ordinary case by the condition

\[ U^\dagger U = 1 = UU^\dagger. \]

(A.15)
The graded trace of a graded matrix $F$ given in the form (A.10) is defined by

$$\text{tr}_g F = \text{tr} \ a - \text{tr} \ b.$$  \hspace{1cm} (A.16)

Here, tr denotes the conventional trace of a matrix of dimension $2L$ containing as elements commuting variables. Since the order of arguments in the definition (A.6) differs from that adopted by Efetov, our convention (A.16) differs in sign from his. We have $\text{tr}_g F^T = \text{tr}_g F$. Moreover, the graded trace of a product of graded matrices is invariant under cyclic permutations. The convention (A.16) implies that for any two graded vectors $\phi_1$ and $\phi_2$ we have

$$(\phi_1 \cdot \phi_2) = \text{tr}_g [\phi_2 \cdot \phi_1],$$

$$(\phi_1^T \cdot \phi_2) = \text{tr}_g [\phi_2 \cdot \phi_1^T],$$  \hspace{1cm} (A.17)

where the arguments of the graded traces are dyadic products.

Functions of graded matrices are defined in terms of their power-series expansions. The graded determinant of a graded matrix $F$ is defined by

$$\text{det}_g(F) = \exp\{\text{tr}_g \ln(F)\}. \hspace{1cm} (A.18)$$

With $F$ given by eq. (A.10), explicit calculation shows that

$$\text{det}_g(F) = \text{det}(a - \sigma b^{-1} \rho) \cdot (\text{det} \ b)^{-1} \hspace{1cm} (A.19)$$

where on the r.h.s. we deal with ordinary determinants of $2L \times 2L$ matrices containing commuting elements. We have [21] $\text{det}_g(F_1 \cdot F_2) = \text{det}_g(F_1) \cdot \text{det}_g(F_2)$ and $\text{det}_g(F^T) = \text{det}_g(F)$.

**Appendix B. Gaussian integrals involving anticommuting variables**

Integrals over anticommuting variables are defined by

$$\int d\chi = 0 = \int d\chi^* \quad \text{and} \quad \int \chi \ d\chi = (2\pi)^{-1/2} = \int \chi^* \ d\chi^*. \hspace{1cm} (B.1)$$

We use the convention that differentials of anticommuting variables anticommute with each other as well as with the anticommuting variables themselves. Thus

$$\int \chi_1 \chi_2 \ d\chi_1 \ d\chi_2 = -\left( \int \chi_1 \ d\chi_1 \right) \left( \int \chi_2 \ d\chi_2 \right) = -(2\pi)^{-1}. \hspace{1cm} (B.2)$$

The choice of the constant $(2\pi)^{-1/2}$ in eq. (B.1) is arbitrary and introduced for convenience.

Let $A$ be an $N \times N$ matrix with commuting elements and with non-zero eigenvalues. The conventions (B.1) and (B.2) imply
\[
\int \exp \left\{ +i \sum_{k,l=1}^{N} X_k^* A_{kl} X_l \right\} d\chi_1^* d\chi_1 \cdots d\chi_N^* d\chi_N = \det(A/(2i\pi)) .
\]  

(B.3)

We note that the order of integration variables and differentials is important. We define

\[
d[\chi] = d\chi_1^* d\chi_1 \cdots d\chi_N^* d\chi_N
\]

and keep this order for all that follows.

We recall that for ordinary real variables \( S_1, \ldots, S_N \) we have

\[
\int dS_1 \cdots \int dS_N \exp \left\{ \frac{1}{2} \sum_{k,l=1}^{N} S_k A_{kl} S_l \right\} = [\det(A/(2i\pi))]^{-1/2} .
\]

(B.5)

Equations (B.3) and (B.5) can be combined as follows. We introduce the real variables \( S_1^1, \ldots, S_N^1 \) and \( S_1^2, \ldots, S_N^2 \). We write

\[
d[S] = dS_1^1 \cdots dS_N^1 dS_1^2 \cdots dS_N^2 .
\]

(B.6)

We introduce the graded vector \( \phi \) as in eq. (A.6) with

\[
\phi^T = (S_1^1, \ldots, S_N^1; S_1^2, \ldots, S_N^2; X_1, \ldots, X_N; X_1^*, \ldots, X_N^*)
\]

and the graded \( 4N \times 4N \) matrix \( M \) with

\[
M = \begin{pmatrix}
A & A \\
\emptyset & A & A
\end{pmatrix} .
\]

(B.8)

Then,

\[
\int d[S] d[\chi] \exp \left\{ \pm \frac{1}{2} i(\phi^* M \phi) \right\} = +1
\]

(B.9)

combines eqs. (B.3) and (B.5). Equation (B.9) is the basis of the use of graded vectors and matrices in the present context, since it shows that the generating function \( Z \) defined in eq. (3.2) is, in the absence of source terms, normalized to unity.

The fundamental formula (B.9) may be viewed as a special case of Wegner's integral theorem [17]. In this theorem, graded symmetries are used to show that integrals over invariant functions equal the integrand taken at argument zero, cf. also appendix L.

In the sequel, we use the convention

\[
\]

(B.10)

Under a transformation of variables \( \phi = \phi(\phi') \), the volume element \( d[\phi] \) is replaced by
\( \detg(\partial \phi / \partial \phi') d[\phi'] \). Here, \( \partial \phi / \partial \phi' \) is a matrix of dimension \( 4N \) consisting of the partial derivatives of the variables in \( \phi \) with respect to the variables in \( \phi' \) [11].

Next we consider the integral

\[
I = \int d\chi_1 \cdots d\chi_N \exp \left\{ \frac{1}{2} \sum_{k,l} A_{kl} \chi_k \chi_l \right\}. \tag{B.11}
\]

Without loss of generality, we may assume the matrix \( A \) to be skew-symmetric. To calculate \( I \), we write

\[
I^2 = \int d\chi_1^1 \cdots d\chi_N^1 d\chi_1^2 \cdots d\chi_N^2 \exp \left\{ \frac{1}{2} \sum_{k,l} A_{kl} \left( \chi_k^1 \chi_l^1 + \chi_k^2 \chi_l^2 \right) \right\}. \tag{B.12}
\]

The form of the exponent suggests introducing new variables,

\[
\chi_k^1 = \frac{1}{\sqrt{2}} (\theta_k + \theta_k^*); \quad \chi_k^2 = \frac{i}{\sqrt{2}} (\theta_k - \theta_k^*). \tag{B.13}
\]

This transformation implies \( d\chi_k^1 d\chi_k^2 = i \ d\theta_k^* d\theta_k \). Moreover, rearranging factors in the volume element we have

\[
d\chi_1^1 \cdots d\chi_N^1 d\chi_1^2 \cdots d\chi_N^2 = (-)^{N(N-1)/2} d\chi_1^1 d\chi_1^2 \cdots d\chi_N^1 d\chi_N^2.
\]

It follows that

\[
I^2 = i^N (-)^{N(N-1)/2} \int d[\theta] \exp \left\{ \sum_{k,l=1}^{N} \theta_k^* A_{kl} \theta_l \right\} = i^N (-)^{N(N-1)/2} \det (-A/(2\pi)), \tag{B.14}
\]

where we have used eq. (B.3). For skew-symmetric matrices \( A \), \( \det A = 0 \) unless \( N \) is even. But for even \( N \), we have \( i^N (-)^{N(N-1)/2} = +1 \) and \( \det(-A) = \det A \) and therefore

\[
I^2 = \det(A/(2\pi)) \quad \text{or} \quad I = [\det(A/(2\pi))]^{1/2}. \tag{B.15}
\]

**Appendix C. Evaluation of the integral (4.9) over \( d[\psi] \)**

In the expression \( i \psi_{\mu} (L^{1/2}NL^{1/2})_{\mu \nu, \alpha \beta} \psi_{\nu} \) which appears in the exponent with \( 1 \leq \mu, \nu \leq N \) and \( 1 \leq \alpha, \beta \leq 8 \) we interchange in \( \psi_{\mu} \) the indices \( \alpha = 5 \) and \( \alpha = 6 \), and the indices \( \alpha = 7 \) and \( \alpha = 8 \). We proceed likewise with the \( \alpha \)-indices in the matrix \( i(L^{1/2}NL^{1/2})_{\mu \nu, \alpha \beta} \). In the resulting expressions for \( \psi_{\mu}^\dagger \) and \( i(L^{1/2}NL^{1/2})_{\mu \nu, \alpha \beta} \) we multiply all terms carrying the (new) indices \( \alpha = 5 \) and \( \alpha = 7 \) with minus one. The result is that \( \psi_{\mu}^\dagger \rightarrow \psi_{\mu}^* \) (this follows from the definitions given in appendix A). The resulting matrix is denoted by \( i(L^{1/2}NL^{1/2})_{\mu \nu, \alpha \beta} \). We decompose it into \( 4N \times 4N \) matrices by writing

\[
i(L^{1/2}NL^{1/2}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{C.1}
\]
The symmetry of the graded vector $\psi$ implies that $A^T = A$, $D^T = -D$ and $B^T = -C$.

We relabel the integration variables by introducing

$$\{T_{\mu\alpha}\} = \{S^\mu_1(1), S^\mu_2(1), S^\mu_3(2), S^\mu_4(2)\} \quad \text{with} \quad \mu = 1, \ldots, N; \alpha = 1, \ldots, 4$$

and

$$d[T] = \prod_{i=1}^2 \prod_{p=1}^N \prod_{\mu=1}^4 dS^\mu_i(p),$$

and likewise for the anticommuting variables

$$\{\tau_{\mu\alpha}\} = \{\chi^\mu_1(1), \chi^\mu_2(1), \chi^\mu_3(2), \chi^\mu_4(2)\} \quad \text{with} \quad d[\tau] = \prod_{\mu=1}^N (d\tau_{\mu,1} d\tau_{\mu,2} d\tau_{\mu,3} d\tau_{\mu,4}).$$

We recall the definition (B.4) of the differentials appearing in $d[\psi]$ and conclude that

$$d[\psi] = d[T] d[\tau]. \quad (C.2)$$

The integral takes the form

$$\int d[T] d[\tau] \exp \left\{ \frac{1}{2} (T^T A T + T^T B \tau + \tau^T C T + \tau^T D \tau) \right\}. \quad (C.3)$$

We shift the $\tau$-variables, $\tau \rightarrow \tau - D^{-1} C T$. As shown in ref. [11], such a shift does not affect the value of the integral. This is an elementary property of integrals over Grassmann variables. The transposed vector $\tau^T$ is shifted accordingly, $\tau^T \rightarrow \tau^T - T^T C T (D^{-1})^T$. We now use that $D^T = -D$ and $C^T = -B$ to obtain $\tau^T \rightarrow \tau^T - T^T B D^{-1}$. With these substitutions, the integral (C.3) takes the form

$$\int d[T] \exp \left\{ \frac{1}{2} T^T (A - B D^{-1} C) T \right\} \int d[\tau] \exp \left\{ \frac{1}{2} \tau^T D \tau \right\}. \quad (C.4)$$

The first integral is evaluated using eq. (B.5), the second one, using eqs. (B.11) and (B.15). The result is

$$\detg(iL^{1/2}N L^{1/2})^{-1/2}. \quad \text{We now retrace the steps described at the beginning of this appendix and use that} \quad \detg(iL) = 1. \quad \text{The result is \detg(N)}^{-1/2} \quad \text{since the value of \detg remains unchanged under a simultaneous interchange of two rows, and a multiplication of one of the rows with minus one. We finally use eq. (A.18) and obtain eq. (4.12).}$$

**Appendix D. Symmetry properties, and the parametrization of variable transformations: Compactification**

We construct the group of transformations of the original integration variables $\psi^T_\mu = (S^\mu_i(p), \chi^\mu_\alpha(p), \chi^\mu_\alpha(p))$ which leave the form (5.1) invariant. For every $\mu$, we consider the linear transformations
Here, $\hat{T}$ is a graded $8 \times 8$ matrix. We require (i) that the form (5.1) be invariant, and (ii) that the $\psi^{\mu\alpha}$ have the same symmetry property as the $\psi_{\mu\alpha}$. Condition (i) yields

$$\hat{T}^*L\hat{T} = L.$$  \hspace{1cm} \text{(D.2)}

With $L^2 = 1$ this implies that every $\hat{T}$ has an inverse $\hat{T}^{-1} = L\hat{T}^*L$. As for condition (ii), we observe that the $S_\mu^x(p)$ are real, and that $(\chi_\mu(p))^* = \chi_\mu^*(p)$. Formally

$$\psi^{\mu\alpha} = \sum_{\beta = 1}^8 C_{\alpha\beta} \psi^{\mu\beta}$$  \hspace{1cm} \text{(D.3)}

where $C$ is a graded $8 \times 8$ matrix. In $2 \times 2$ block matrix notation, $C$ has the form

$$C = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix};$$  \hspace{1cm} \text{(D.4)}

the $2 \times 2$ matrix $\gamma$ is given by

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} \text{(D.5)}

We note that $C^{-1} = C^T$. Condition (ii) then reads $\psi^{\mu\alpha} = \sum_{\beta = 1}^8 C_{\alpha\beta} \psi^{\mu\beta}$, or

$$\hat{T} = C\hat{T}^*C^T.$$  \hspace{1cm} \text{(D.6)}

Conditions (D.3) and (D.6) constitute the extension of the concept of "reality" to graded vectors and matrices.

With two matrices $\hat{T}_1$ and $\hat{T}_2$ each obeying eqs. (D.2) and (D.6), the product $\hat{T}_1 \cdot \hat{T}_2$ also obeys these equations. This and the existence of an inverse for every matrix $\hat{T}$ obeying eq. (D.2) show that the matrices $\hat{T}$ form a group. This group was referred to as $G$ in section 5, and is a graded Lie group denoted by UOSP (2, 2/2, 2). In words: A pseudo-unitary orthosymplectic Lie group. The adjective pseudo-unitary derives from eq. (D.2): For $L = 1$, the matrices $\hat{T}$ would be unitary. The adjective orthosymplectic refers to the mixed orthogonal-symplectic features induced by the matrix $C$ in eq. (D.6). The group $G$ encompasses two subgroups that deserve particular mention. Consider matrices $R_1$ ($R_2$) which induce transformations only among the variables with $p = 1$ ($p = 2$), respectively. These sets of matrices, subject to conditions (D.2) and (D.6), each form a genuinely unitary subgroup UOSP (2/2). (Unitarity follows from eq. (D.2) if we recall that $L_{\alpha\beta} = \delta_{\alpha\beta}(-)^{p+1}$. The numerals (2/2) indicate that the transformations act on a graded vector space with 2 commuting and 2 anticommuting components.)

It is instructive to construct the generators $G$ of $R_1$ (or of $R_2$). For this purpose, we switch notation. We abandon the Boson–Fermion notation used in table 4.2 and up to section 5.4 included, and we use
instead the \([1, 2]\) block notation of table 4.3. In this notation, \(R_1\) contains a non-trivial \(4 \times 4\) graded matrix in the \((1, 1)\) block, the unit matrix in the \((2, 2)\) block, and zeroes in the \((1, 2)\) and \((2, 1)\) blocks. For \(R_2\), the roles of 1 and 2 are interchanged. Writing \(R_1 = 1 + G_1\) we obtain for \(G_1\) infinitesimal,

\[
G_1 = CG_1^* C^T, \\
G_1^* = -G_1.
\]

The \((1, 1)\) block of \(G_1\) has the form given in table D.1. The variables \(\varphi\) and \(m\) are real and commuting, the variable \(m_1\) is complex and commuting, and the variables \(\alpha_1, \alpha_1^*, \alpha_2, \alpha_2^*\) are anticommuting. The first \(2 \times 2\) block of \(G_1\) contains the generator of an orthogonal transformation \(O(2)\). This is due to the appearance of the term \(\Sigma_{\mu=1}^4 S_{\mu}^\dagger(p) S_{\mu}(p)\) for \(p = 1\) in eq. (5.1), which is orthogonally invariant. Similarly, the fourth \(2 \times 2\) block of \(G_1\) contains the three generators of an \(SU(2)\) group, related to the appearance of the form \((\chi_{\mu}^*(p) \chi_{\mu}(p) + \chi_{\mu}^*(p) \chi_{\mu}(p))\) for \(p = 1\) in eq. (5.1). [Indeed, for the generators \(g\) of \(SU(2)\) the condition \(UU^\dagger = 1\) yields \(g^\dagger = -g\), and the condition \(\det U = 1\) yields \(\text{tr } g = 0\) which together result in the form of the fourth \(2 \times 2\) block of \(G_1\) in table D.1.] Finally, the anti-Hermitean character of \(G_1\) with respect to the Grassmann variables \(\alpha_i, \alpha_i^*, i = 1, 2\), shows that these variables induce unitary transformations among the \(S_{\mu}^\dagger(1)\) and \(\chi_{\mu}(1), \chi_{\mu}^*(1)\), owing to the unitary symmetry of the quadratic form (5.1).

We combine the unitary transformations \(R_1\) and \(R_2\) into a single group of matrices \(R\), the direct product \(UOSP(2/2) \otimes UOSP(2/2)\). The transformations \(R\) leave the saddle point \(\sigma_0^\dagger\) in eq. (5.38) invariant. We are mainly interested in transformations connecting the variables denoted by \(p = 1\) with those denoted by \(p = 2\). Indeed, these transformations are not encountered in the study of the one-point function, and they constitute the \textit{novel} element in the two-point function. With this in mind, we first write down the generators \(G\) of a general transformation \(T\) satisfying eqs. (D.2) and (D.6), using \([1, 2]\) block notation. In the \((1, 1)\) and \((2, 2)\) blocks, we retrieve, of course, the conditions (D.7) and the form of table D.1. Denoting the remaining generators in the \((1, 2)\) and \((2, 1)\) blocks by \(G_{12}\) and \(G_{21}\), respectively, we find that (we recall the definition (A.14))

\[
G_{12} = CG_{12}^* C^T; \quad G_{21} = CG_{21}^* C^T; \quad G_{12}^* = +G_{21}.
\]

Here, \(C\) is a graded \(4 \times 4\) matrix defined via block construction by the two \(2 \times 2\) matrices 1 and \(\gamma\). The crucial point in eq. (D.8) is the plus sign in the last relation which differs from the corresponding sign in eq. (D.7) and is caused by the form of the matrix \(L\). This sign shows that \(G\) is \textit{Hermitean} in the \((1, 2)\) and \((2, 1)\) block, and it therefore induces a \textit{non-compact} transformation between all the variables with \(p = 1\) and those with \(p = 2\). This is best seen by writing down a special set of transformations, generated by \(G_{21}\) and \(G_{12}\) in the absence of both \(G_1\) and \(G_2\). Such transformations can be written as
We have used [1, 2] block notation. The form of the matrix \( t' \) is given in table D.2 with \( a, b, c, d \) real and commuting, \( z \) and \( w \) complex and commuting, and \( \eta_1, \rho_1, \eta_1^* \) (where \( i = 1, 2 \)) anticommuting. We see immediately that \( t \) and \( t' \) obey conditions (D.8). [It is also straightforward to check directly that \( \hat{T}_0 \) obeys the conditions (D.2) and (D.6).] The non-compactness is finally evidenced by the plus sign under the square roots in eq. (D.9): The matrix \( \hat{T}_0 \) is the generalization of the transformations (5.2) and (5.3) to several variables, some of them complex. The matrices \( \hat{T}_0 \) parametrize the coset space \( \text{UOSP} (2, 2/2, 2)/[\text{UOSP} (2/2) \otimes \text{UOSP} (2/2)] \).

The general decomposition of the matrices \( \hat{T} \) in terms of the matrices \( \hat{T}_0 \) from this coset space and of the matrices \( R \) in \( \text{UOSP} (2/2) \otimes \text{UOSP} (2/2) \) is obtained as follows. From the form of the generators \( G \) of \( \hat{T} \) we see that every matrix \( \hat{T} \) can be written as

\[
\hat{T} = R^t \hat{T}_0 R = (R^t R) \hat{T}_0 R^t,
\]

with \( R \) and \( R' \) in \( \text{UOSP} (2/2) \otimes \text{UOSP} (2/2) \) and \( \hat{T}_0 \) of the form (D.9). Writing

\[
R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}
\]

we see that \( R^t \hat{T}_0 R \) again has the form (D.9), with \( t \) replaced by \( R^t_1 t R_1 \). It follows that all matrices \( \hat{T} \) can be written as

\[
\hat{T} = R \hat{T}_0,
\]

with \( \hat{T}_0 \) of the form (D.9) and \( R \in \text{UOSP} (2/2) \otimes \text{UOSP} (2/2) \).

Under the transformations \( \hat{T} \) of eq. (D.10b), the matrix \( A \) defined in eq. (4.3) transforms into \( A' \) with

\[
A' = (L^{1/2} \hat{T} L^{-1/2}) A (L^{-1/2} \hat{T}^t L^{1/2}) = T A T^{-1},
\]

where we have used that \( L^2 = 1 \). This shows that a transformation of variables (D.1) induces on \( \sigma \) the transformation

\[
\sigma \rightarrow T^{-1} \sigma T, \quad T = L^{1/2} \hat{T} L^{-1/2}.
\]

With eqs. (D.10) and (D.9), eq. (D.12) implies
\[ T = R \begin{pmatrix} (1 + t^t t)^{1/2} & i t^t \\ -i t & (1 + t^t)^{1/2} \end{pmatrix} \]  

(D.13)

where we have chosen \((L^{1/2})_{\alpha \beta} = \delta_{\alpha \beta} i^{1-\rho} \).

Unfortunately, the transformations (D.12) are non-compact with respect to both \(\sigma_B\) and \(\sigma_F\). This feature, while desirable for \(\sigma_B\), is at variance with the convergence- and saddle-point properties discussed in sections 5.2 and 5.3 for \(\sigma_F\), and must be remedied. This is accomplished by making the substitutions

\[ t^t \rightarrow t_{12}, \quad t \rightarrow t_{21} \]  

(D.14)
in eq. (D.9), with \(t_{12}\) and \(t_{21}\) given in table D.3. While the variables \(a, b, c, d\) remain unchanged under the substitution (D.14), the variables \(z, w\) (and, likewise the anticommuting variables \(\eta_1, \rho_1, \eta_1^*, \rho_1^*\)) are multiplied by \(i\). This makes the generator of the modified matrix anti-Hermitean in \(z, w\), and this is what is needed: The parametrization of \(\sigma\) is compact with respect to \(\sigma_F\). (For the anticommuting variables, compactness and non-compactness are not meaningful concepts. For details, see section 8 and appendices I and K.) The substitution \((z, w) \rightarrow (iz, iw)\) is the generalization of the substitution \(\sinh \beta \rightarrow i \sin \beta\) in section 5.2. We show in appendix I that the “reality” condition on \(T\) (the second of eqs. (D.17) below) restricts the range of \(z\) and \(w\) to \(|z|^2 + |w|^2 \leq 1\), see eq. (1.19). This is the formal expression for the compactification.

The transformation (D.14) can formally be written as

\[ t_{12} = t^t k^{-1/2}, \quad t_{21} = k^{-1/2} t, \]  

(D.15)

where \(k^{1/2}\) is a diagonal \(4 \times 4\) matrix with elements \((1, 1, -i, -i)\). We define the graded \(8 \times 8\) matrix \(K\) as

\[ K = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}. \]  

(D.16)

While the “compactified” matrix \(\hat{T}_{0c}\), obtained from \(\hat{T}_0\) of eq. (D.9) with the substitutions (D.14), obviously does not obey eqs. (D.2) and (D.6), it can be checked by explicit calculation that we have

\[ \hat{T}_{0c}^* KL \hat{T}_{0c} = KL; \quad \hat{T}_{0c} = (CK) \hat{T}_{0c}^* (C^T K). \]  

(D.17)

Obviously, eqs. (D.17) are not consistent with the properties (D.2) and (D.6) obeyed by the transformations \(R_1\) and \(R_2\); the latter should therefore be reconstructed in such a way that they also obey
eqs. (D.17) instead. [This change obviously affects only \( R_2 \), cf. eq. (D.16).] Calling the resulting transformation matrices \( R_{1c} \), \( R_{2c} \) and their product \( R_c \), we see that all “compactified” transformations \( \hat{T}_c \), defined by conditions (D.17), form a group and have the general form

\[
\hat{T}_c = R_c \hat{T}_{0c}.
\]

The transformations \( T_c \) thereby induced on \( \sigma \) (see eqs. (D.12)) can be written as

\[
T_c = R_c \left( \begin{array}{cc}
(1 + t_{12}t_{21})^{1/2} & it_{12} \\
-it_{21} & (1 + t_{21}t_{12})^{1/2}
\end{array} \right). \tag{D.19}
\]

These are the transformations used to parametrize \( \sigma \). The compactification made in the present appendix may appear somewhat ad hoc. We therefore give another derivation, starting from first principles, in appendix E.

Appendix E. Symmetry properties, and the parametrization of variable transformations:

Exact approach

In this appendix, we remedy the unsatisfactory feature introduced at the end of appendix D, where we used a somewhat ad-hoc prescription to “compactify” the transformations \( \hat{T} \) with respect to the Fermion–Fermion variables. We do this by utilizing the freedom of choice explained in section 5.4. Formally, this freedom of choice is expressed by the following version of eq. (B.3):

\[
\int d[\chi] \exp\left\{ c \sum_{k, l=1}^{N} \chi^{t}_k A_{kl} \chi_l \right\} = \det(cA/(-2\pi)). \tag{E.1}
\]

Equation (E.1) is valid for any complex number \( c \). Equation (B.3) was used to obtain eq. (3.4) for \( Z(E, J) \). We now query how this and subsequent equations are modified if we use eq. (E.1) with \( c = c_1 \) for \( Z(E(1), J(1)) \) and with \( c = c_2 \) for \( Z^*(E(2), J(2)) \), and how best to use the freedom of choice of \( c_1 \) and \( c_2 \) to obtain a formulation in which an ad-hoc “compactification” is not needed.

The indefinite metric for the \( \chi_{\mu}(p) \) in the form (5.1) is caused by the fact that in section 3 we made the choice \( c_1 = +i \), \( c_2 = -i \). The convergence arguments of section 5.2 show the necessity of having a definite metric for the \( \chi_{\mu}(p) \). This is achieved by choosing \( c_1 = c_2 \). Finally, it is convenient to have a “metric tensor” which is real. This is accomplished by putting \( c_1 = c_2 = \pm i \). We choose the positive sign.

A second arbitrariness in the choice of phase factors resides in the definition of the graded vector \( \psi \), eq. (3.10). Since \( \psi \) occurs only in quadratic forms involving both \( \psi \) and \( \psi^* \), see eqs. (B.9) or (3.15), we are at liberty to replace in the defining equation (3.10) \( \chi_{\mu}^t \) by \( i\chi_{\mu} \) and \( \chi_{\mu} \) by \( i\chi_{\mu}^* \). Making the corresponding replacement in \( \psi^t \) (with \( i \rightarrow -i \)), we obviously do not change the content of any of the equations leading to eq. (4.12). We are thus free to consider, instead of the vector \( \psi_{\mu\alpha} \), another vector \( \hat{\psi}_{\mu\alpha} \) in which the above-mentioned substitution has been made for the \( \chi \)-variables with \( p = 2 \) only. Since

\[
\begin{pmatrix}
(i\chi_{\mu}^t) \\
(i\chi_{\mu})^t
\end{pmatrix} = -\gamma \begin{pmatrix}
(i\chi_{\mu})^* \\\n(i\chi_{\mu}^t)
\end{pmatrix}, \tag{E.2}
\]
we now have the relation
\[ \hat{\psi} = KC\hat{\psi}^* \] (E.3)
instead of eq. (D.3). The matrix \( K \) is defined in eq. (D.16).

We now implement these two modifications in the theory, starting at the very beginning, section 3. The change in the definition of \( \psi (\psi \rightarrow \hat{\psi}) \) actually does not affect any of the derivations, and we focus attention on the phase factors introduced by putting \( c_1 = c_2 = +i \). Without giving details, we merely indicate which of the formulas leading to eq. (4.12) have to be modified. The expression (3.14) for \( Z(E(1), E(2), J) \) remains formally unchanged, and so do the derivative formulas (3.16) as well as the definitions (3.11) to (3.13) and (3.18) and (3.19), if we replace in eq. (3.15) the matrix \( L^{1/2} \) by the matrix \( (L^{1/2}K^{1/2}) \), with \( K = \{ \delta_{\mu\nu}K_{ab} \} \) and \( K \) defined in eq. (D.16), and add the term \( i\pi \xi \) to the r.h.s. of eq. (3.15). Equation (4.2) requires the substitutions \( L^{1/2} \rightarrow L^{1/2}K^{1/2} \), \( L \rightarrow LK \). The matrix \( A \) of eq. (4.3) now takes the form
\[ A_{\alpha\beta} = i\lambda [(KL)^{1/2}]_{\alpha\gamma} \hat{\psi}_{\mu\gamma} \hat{\psi}^*_{\mu\delta} [(KL)^{1/2}]_{\delta\beta}, \] (E.4)
and the fundamental form which we require to be invariant under transformations of \( \hat{\psi} \) is now
\[ \hat{\psi}_{\mu\alpha} (KL)_{\alpha\beta} \hat{\psi}_{\nu\beta} = \sum_p (-)^{p+1} \left[ \sum_i S'_\mu(p) S'_\nu(p) \right] + \sum_p (\chi^s_\mu(p) \chi^s_\nu(p) + \chi^x_\mu(p) \chi^x_\nu(p)). \] (E.5)

Equation (4.12) and the definitions (4.10) and (4.11) remain formally unchanged. [The additional term \( i\pi \xi \) on the r.h.s. of eq. (3.15) is cancelled by a factor \( \left[ \text{det} (iKL) \right]^{-1/2} \) which arises as one repeats the steps in appendix C at the very end of the derivation.] In conclusion, our main result, eq. (4.12), is unchanged, but the relevant quadratic form is now that of eq. (E.5) and not that of eq. (5.1). This explicitly demonstrates that the metric with respect to the anticommuting variables is arbitrary. Applying the steps in appendix D to the form (E.5), and to a vector \( \hat{\psi} \) which behaves under complex conjugation for the anticommuting variables with \( p = 2 \) as in eq. (E.3), we are led to eqs. (D.17) as the defining conditions for the transformation matrices \( \hat{T}_c \). These transformation matrices can then be written in the form (D.18), with \( R_c \) a block diagonal matrix \( \left( \begin{array}{cc} R^t_1 & 0 \\ 0 & R^t_2 \end{array} \right) \), with both \( R^t_1 \) and \( R^t_2 \) subject to eqs. (D.17), and with \( \hat{T}_ac \) given by
\[ \hat{T}_ac = \left( \begin{array}{cc} 1 + t_{12}t_{21}^{1/2} & t_{12}^{1/2} \\ t_{21} & (1 + t_{21}t_{12})^{1/2} \end{array} \right). \] (E.6)

The matrices \( t_{12} \) and \( t_{21} \) are listed in table D.3. The matrices \( R^t_1 \) and \( R^t_2 \) are needed for the construction of the variables \( P_{11} \) and \( P_{22} \), respectively, cf. eqs. (5.32). It was remarked already that eqs. (D.17) yield for \( R^t_1 \) ordinary unitarity conditions as well as \( R^t_1 = CR^t_1C^T \). Thus, the generators of \( R^t_1 \) are those of table D.1. For \( R^t_2 \), the conditions read \( R^t_2KR^t_2 = K \) and \( R^t_2 = CKR^t_2C^TK \). The generators \( G_2 \) of these matrices are displayed in table E.1. The notation for real/complex and commuting/anticommuting variables is the same as in table D.1. We note that the generators in tables D.1 and E.1 differ only in the phase factors of the anticommuting variables.

Returning to eq. (E.4) we observe that the transformations \( \hat{T}_c \) acting on \( \hat{\psi} \) induce on \( \sigma \) the transformation \( \sigma \rightarrow \hat{T}_c^{-1}\sigma \hat{T}_c \), with
Table E.1
The generators of the pseudo-unitary transformations $R_2$ subject to eqs. (D.17)

|   | $\psi$ | $i\sigma_1$ | $i\sigma_1^\dagger$ | $-\psi$ | $0$ | $i\sigma_2$ | $i\sigma_2^\dagger$ | $-i\sigma_1$ | $-i\sigma_2$ | $m^\dagger$ | $-i\sigma_1^\dagger$ | $i\sigma_2^\dagger$ | $-m$ | $-i\sigma_1$ | $i\sigma_2$ | $m$ | $i\sigma_1^\dagger$ | $i\sigma_2^\dagger$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

$$T = (KL)^{1/2} \tilde{T}_c (KL)^{-1/2}.$$  

This equation differs from eq. (D.12), and the matrices $T$ and $\tilde{T}$ are obviously related by

$$\tilde{T} = K^{1/2}T K^{-1/2}.$$  

This seems to suggest that the parametrizations of $\sigma$ obtained via appendix D and via appendix E are, after all, not equivalent, although the matrices $\tilde{T}_c$ are the same in both approaches. This, however, is not the case. With the help of eq. (6.7), and the defining equations (5.41) and (5.39) for $\sigma_c$ and $\delta P$ it is easily checked that the effective Lagrangian $\mathcal{L}_{\text{eff}}$ in eq. (6.6) is invariant under the substitution $T \rightarrow \tilde{T}$. This shows that the approaches in appendices D and E do yield identical results, at least in the limit $N \rightarrow \infty$. In the main body of the paper, we use the parametrization derived in appendix D, i.e., we work with the matrices $T$ rather than the matrices $\tilde{T}$, except for the end of appendix I where we found it simpler to use the formulation of appendix E.

**Appendix F. The Jacobian of the transformation (5.30)**

We write this transformation in the form

$$\sigma = T_0^{-1}PT_0, \quad (F.1)$$

where $P = R^{-1}\sigma_\alpha R$ is block-diagonal in [1, 2] block notation. We view eq. (F.1) as a transformation leading from a set of variables $\{\sigma\}$ with associated volume element $d[\sigma]$ (cf. section 4) to another set of variables $\{P, t_{12}\}$. This latter set is used in carrying out the integrations in sections 6 and 8. We calculate part of the Jacobian of this transformation. Variation of eq. (F.1) yields

$$\delta \sigma = T_0^{-1}[\delta P - [(\delta T_0)T_0^{-1}, P]]T_0. \quad (F.2)$$

where we have used that $(\delta T_0^{-1})T_0 = -T_0^{-1}\delta T_0$. The factors $T_0$ and $T_0^{-1}$ multiplying the curly bracket do not affect the value of the Jacobian and can be omitted. Doing so, and recalling that $P$ is block-diagonal, we show in table F.1 in block notation in which way the three blocks $\delta \sigma_{11}$, $\delta \sigma_{22}$ and $\delta \sigma_{12}$ depend on the variables on the right-hand side (shaded areas indicate non-vanishing terms), with $\delta T' = (\delta T_0)T_0^{-1}$.

In setting up the Jacobian, one encounters precisely the matrix structure shown in table F.1. Because of the various blocks involving zeroes, the Jacobian is the product of the three factors $\det g(\delta \sigma_{11}/\delta P_{11})$, $\det g(\delta \sigma_{22}/\delta P_{22})$, and $\det g(\delta \sigma_{12}/\delta T'_{12})$. The form of eq. (F.2) shows that $\det g(\delta \sigma_{11}/\delta P_{11}) = 1$ and
Table F.1
Dependence of $\delta \sigma$ on $\delta P$ and $\delta T'$

<table>
<thead>
<tr>
<th>$\delta \sigma_{11}$</th>
<th>$\delta P_{11}$</th>
<th>$\delta P_{22}$</th>
<th>$\delta T'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0</td>
<td>0</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>0</td>
</tr>
</tbody>
</table>

$\det g(\delta \sigma_{22}/\delta P_{22}) = 1$. We are left with

$$ \delta \sigma_{12} = -([\delta T', P])_{12}. \tag{F.3} $$

Without loss of generality, we may take the matrices $P_{11}$ and $P_{22}$ to be diagonal; the unitary or pseudo-unitary transformations $R_1$ and $R_2$ effecting the transformation to diagonal form [see eq. (5.32)] do not change the value of the Jacobian. Denoting the eigenvalues of $P_{11}$ by $(\lambda_1(1), \lambda_2(1), \lambda(1), \lambda(1))$ and those of $P_{22}$ by $(\lambda_1(2), \lambda_2(2), \lambda(2), \lambda(2))$, we find from eq. (F.3)

$$ \mathcal{F}(P) = \det g(\delta \sigma_{12}/\delta T'_{12}) \tag{F.4} $$

$$ = \left| \frac{\left( \lambda_1(1) - \lambda_1(2) \right) \left( \lambda_2(1) - \lambda_2(2) \right) \left( \lambda_3(1) - \lambda_3(2) \right) \left( \lambda_4(1) - \lambda_4(2) \right) \left( \lambda(1) - \lambda(2) \right)^4}{\left( \lambda_1(1) - \lambda(2) \right)^2 \left( \lambda_2(1) - \lambda(2) \right)^2 \left( \lambda(1) - \lambda_1(2) \right)^2 \left( \lambda(2) - \lambda_1(2) \right)^2} \right|. $$

The volume element $d[\sigma]$ takes the form

$$ \mathcal{F}(P) \ d[P] \ d\mu(t) \tag{F.5} $$

where

$$ d\mu(t) = \det g(\delta T'_{12}/\delta t_{12}) \ d[\tau_{12}]. \tag{F.6} $$

The factor $\det g(\delta T'_{12}/\delta t_{12})$ is worked out in appendix K. Equation (F.5) is used in sections 5.5 and 6.

It remains to calculate $\mathcal{F}(P)$ at and near the saddle point. Using

$$ P = \sigma^0_D + \delta P \tag{F.7} $$

near the saddle point, and recalling the substitution $\delta P' = N^{1/2} \delta P$ used in section 6 to carry out the limit $N \to \infty$, we see that except for terms of order $N^{-1/2}$, the eigenvalues of $P$ are given by $\sigma^0_D$. Using eq. (5.38) for $\sigma^0_D$, we find from eq. (F.4) that $\mathcal{F}(P) = 1 + O(N^{-1/2})$. This is used in section 6.

It was claimed at the end of section 5.5 that the two saddle points with $\Re \tau_1 > 0$, $\Re \tau_2 > 0$ and with $\Re \tau_1 < 0$, $\Re \tau_2 < 0$ [cf. eq. (5.26)] give a contribution to the saddle-point integral which vanishes in the limit $N \to \infty$. We briefly indicate why this is so. We consider the choice $\Re \tau_1 > 0$, $\Re \tau_2 > 0$ (the other case can be discussed analogously). To simplify the argument, we also put $E = 0$. The choice just indicated implies that the saddle point differs from that given in eq. (5.38): The (1, 1) block of the diagonal matrix in that equation now reads $-ik$, where $k$ is the $4 \times 4$ graded matrix defined below eq. (D.15). As a consequence, we have $P_{11} = -iR_1^{-1}kR_1$. Without making use of the condition $[\sigma^0_D, \delta P] = 0$ used in the derivation of eq. (6.6), and repeating this derivation, we now find that the effective
Lagrangian for the massive variables takes the form \(-\frac{N}{4} \text{tr}(\delta P_{11})^2 + \frac{N}{4} \text{tr}(P_{11}\delta P_{11}P_{11}\delta P_{11}) + \cdots\). The dotted terms relate to \(\delta P_{22}\) and do not differ from those in eq. (6.6). We introduce \(\delta P_{11} = N^{1/2}(R_{1} \delta P_{11} R_{1}^{-1})\); the effective Lagrangian is then given by \(-\frac{1}{4} \text{tr}(\delta P_{11})^2 - \frac{1}{4} \text{tr}(k\delta P_{11} k \delta P_{11}) + \cdots\). The difference to the previous case is that now the matrix \(k\) appears instead of the unit matrix. As a consequence, the Grassmann-valued elements in \(\delta P_{11}\) drop out of the effective Lagrangian for \(N \to \infty\). Before we can conclude that the integral therefore vanishes we must show that these variables do not reappear in the Jacobian \(\mathcal{F}(P)\). Applying eq. (F.4) to the present choice of the saddle point, we see that there are cancellations in both numerator and denominator. Therefore, we must evaluate the eigenvalues of \(P_{11}\) to higher order in \(N^{-1/2}\). In analogy to eq. (F.7), we have now

\[
R_{1}P_{11}R_{1}^{-1} = -ik + N^{-1/2} \delta P_{11} .
\]

Since \(k\) is in block construction given by \((1, 1, -1, -1)\), it is necessary to use degenerate perturbation theory. This involves only the Boson–Boson and the Fermion–Fermion blocks of \(\delta P_{11}\) but not the Grassmann-valued elements which contribute to the eigenvalues only to the order \(N^{-1}\). Using this result in eq. (F.4) we see that the Grassmann-valued elements in \(\delta P_{11}\) do not occur in \(\mathcal{F}(P)\) as \(N \to \infty\). The integral over the associated variables therefore vanishes in this limit.

### Appendix G. Ward identity and unitarity

The nuclear \(S\)-matrix is unitary, \(SS^* = 1\). In the representation of eq. (2.2), this follows from an identity obeyed by the matrix \(D\),

\[
[D^{-1}(E)]_{\mu\nu} - [D^{-1}(E)]_{\mu\nu} = 2i\pi \sum_{\alpha_1,\alpha_2} [D^{-1}(E)]_{\mu\mu_1} W_{\mu_1\alpha} W_{\nu\alpha} [D^{-1}(E)^*]_{\nu\nu} .
\]

Averaging over the ensemble preserves unitarity, \(SS^* = 1\). In analogy to eq. (G.1), this condition holds provided we have

\[
\overline{[D^{-1}(E)]_{\mu\nu} - [D^{-1}(E)]_{\mu\nu}} = 2i\pi \sum_{\alpha_1,\alpha_2} [D^{-1}(E)]_{\mu\mu_1} W_{\mu_1\alpha} W_{\nu\alpha} [D^{-1}(E)^*]_{\nu\nu} .
\]

Using eqs. (3.16) with \(Z\) replaced by \(\tilde{Z}(E(1), E(2), J)\) and taking for the latter the expression (6.8), we show that the unitarity condition holds, by deriving eq. (G.2). Equation (G.2) has the general form of a Ward identity: It connects a special form of the (average) two-point function with the difference of two (average) one-point functions. The derivation provides a test that the steps leading to eq. (6.8) are correct. For the derivation, we rely on symmetry properties of the integration manifold which are destroyed once we carry out the remaining integrals over all Grassmann variables and some of the commuting variables. Hence, an analytical proof of unitarity based on a Ward identity does not appear possible during later stages of the present theory.

Since unitarity involves the \(D\)-matrices taken at \(E(1) = E(2)\), we consider eq. (6.8) with \(\varepsilon = 0\). Derivation of the Ward identity involves a number of formal steps which are valid even in the absence of the convergence-generating term proportional to \(\tilde{a}\) in eq. (6.8). We therefore put \(\tilde{a} = 0\) and focus attention on the effective Lagrangian for \(\varepsilon = 0\),
\[ \mathcal{L}_{\text{eff}}^{0}(\sigma_G, J) = -\frac{1}{2} \text{trg} \ln(E - \lambda \Sigma_G + iW + J). \] (G.3)

The derivation attains maximum compactness if we do not take the matrix \( J \) to be of the form given in expression (3.12c), but instead allow \( J \) to be a general symmetric \( 8N \times 8N \) graded matrix \( \{J_{\mu\nu}\} \). This procedure is permissible as we have made use of the specific form of \( J \) nowhere in the developments of sections 4 through 6. At the end of this appendix, we show that derivatives of \( Z(E(1), E(2), J) \) involving this general matrix lead to formulas similar to eqs. (3.16) and thus to a connection with S-matrix elements.

We consider a transformation of integration variables \( t_{12} \) induced by a transformation \( T_0 \) of the form (5.29) which we presently take to be infinitesimal. Such a transformation leaves the volume element \( d\mu(t) \) invariant as follows from the derivation given in appendix K. It changes \( \sigma_G \) into \( T_0^{-1} \sigma_G T_0 \). Using the cyclic invariance of the graded trace, and writing \( T_0 \equiv \delta T_0 \), we find that the transformation changes \( \mathcal{L}_{\text{eff}}^{0}(\sigma_G, J) \) into

\[ \mathcal{L}_{\text{eff}}^{0}(\sigma_G, J) = -\frac{1}{2} \text{trg} \ln(E - \lambda \Sigma_G + iW + J + [\delta T_0, (iW + J)]). \] (G.4)

A transformation of integration variables leaves \( \tilde{Z}(E(1), E(2), J) \) invariant. Therefore, the terms linear in \( \delta T_0 \) in the expression for \( \tilde{Z} \) must vanish,

\[ \sum_{\mu\nu, \beta} \left. \frac{\partial \tilde{Z}(E(1), E(2), J)}{\partial J_{\mu\nu\beta}} \right|_{(iW + J)} ([\delta T_0, (iW + J)])_{\mu\nu\beta} = 0. \] (G.5)

Instead of the indices \((\alpha, \beta)\), we introduce \([1, 2]\) block notation with \((p, p')\) running from 1 to 2 and \((k, k')\) from 1 to 4. In eq. (G.5), we separate the terms with different \((p, p')\), using the notation \( J_{\mu\nu\beta} \rightarrow J_{\mu\nu kk'}(p, p') \) etc. We also use the fact that \( \delta T_0 \) contains non-vanishing elements only in the \((1, 2)\) and the \((2, 1)\) blocks, see eq. (5.29). This yields

\[
\begin{align*}
\sum_{\mu\nu kk'} & \frac{\partial \tilde{Z}}{\partial J_{\mu\nu kk'}(1, 1)} (\delta T_0(1, 2) J(2, 1) - J(1, 2) \delta T_0(2, 1))_{\mu\nu kk'} \\
+ & \sum_{\mu\nu kk'} \frac{\partial \tilde{Z}}{\partial J_{\mu\nu kk'}(2, 2)} (\delta T_0(2, 1) J(1, 2) - J(2, 1) \delta T_0(1, 2))_{\mu\nu kk'} \\
- & \sum_{\mu\nu kk'} \frac{\partial \tilde{Z}}{\partial J_{\mu\nu kk'}(1, 2)} (2 \delta T_0(1, 2))_{kk'} i\pi \sum_{a} \sum_{a} W_{\mu a} W_{\nu a} \\
+ & \sum_{\mu\nu kk'} \frac{\partial \tilde{Z}}{\partial J_{\mu\nu kk'}(2, 1)} (2 \delta T_0(2, 1))_{kk'} i\pi \sum_{a} \sum_{a} W_{\mu a} W_{\nu a} + \cdots = 0.
\end{align*}
\] (G.6)

The dots indicate terms proportional to either \( J(1, 1) \) or \( J(2, 2) \) which disappear under the operation leading to eq. (G.9). In deriving eq. (G.6), we have used the definition (3.12b) for \( W \). We now choose \( \delta T_0 \) in accordance with eq. (5.29) and table D.3 as

\[(\delta T_0(1, 2))_{kk'} = i \delta_{kk_0} \delta_{k'k_0}, \quad (\delta T_0(2, 1))_{kk'} = -i \delta_{kk_0} \delta_{k'k_0}. \] (G.7)
where we restrict \( k_0 \) and \( k'_0 \) to \( k_0 = 1 \) or \( 2 \) and \( k'_0 = 1 \) or \( 2 \). (We could similarly restrict \( \delta T_0 \) to the Fermion–Fermion part of table D.3 and obtain the same final result.) Inserting this into eq. (G.6), we obtain

\[
\sum_{\mu \nu k} \frac{\partial \tilde{Z}}{\partial J_{\mu \nu k k_0}(2, 1)} J_{\mu \nu k k_0}(2, 1) - \sum_{\mu \nu k} \frac{\partial \tilde{Z}}{\partial J_{\mu \nu k_0 k}(1, 1)} J_{\mu \nu k_0 k}(1, 1)
\]
\[+ \sum_{\mu \nu k} \frac{\partial \tilde{Z}}{\partial J_{\mu \nu k k_0}(2, 2)} J_{\mu \nu k k_0}(2, 2) - \sum_{\mu \nu k} \frac{\partial \tilde{Z}}{\partial J_{\mu \nu k_0 k}(1, 2)} J_{\mu \nu k_0 k}(1, 2)\]
\[- 2i \pi \sum_{\mu \nu a} W_{\mu a} W_{\nu a} \frac{\partial \tilde{Z}}{\partial J_{\mu \nu k_0 k_0}(1, 1)} \]
\[= \sum_{\mu \nu a} W_{\mu a} \frac{\partial \tilde{Z}}{\partial J_{\mu \nu k_0 k_0}(2, 1)} \]
\[+ \ldots . \tag{G.8}\]

We differentiate eq. (G.8) with respect to \( J_{\mu \nu k_0 k_0}(1, 2) \) and put \( J = 0 \). This leads to the disappearance of the dotted terms, and to the result

\[
2 \left[ \frac{\partial \tilde{Z}(E(1), E(2), J)}{\partial J_{\mu \nu ' k_0'}(1, 1)} \right]_{J=0} \delta_{k_0 l} - 2 \left[ \frac{\partial \tilde{Z}(E(1), E(2), J)}{\partial J_{\mu \nu ' k_0'}(2, 2)} \right]_{J=0} \delta_{k_0 l}
\]
\[= 2i \pi \sum_{\mu \nu a} W_{\mu a} W_{\nu a} \frac{\partial^2 \tilde{Z}(E(1), E(2), J)}{\partial J_{\mu \nu k_0 k_0}(1, 2)} \frac{\partial J_{\mu \nu ' k_0'}(2, 2)}{\partial J_{\mu \nu k_0 k_0}(1, 2)} \]
\[= 2i \pi \sum_{\mu \nu a} W_{\mu a} W_{\nu a} \frac{\partial^2 \tilde{Z}(E(1), E(2), J)}{\partial J_{\mu \nu k_0 k_0}(2, 1)} \frac{\partial J_{\mu \nu ' k_0'}(1, 2)}{\partial J_{\mu \nu k_0 k_0}(2, 1)} \] \( J=0 \). \tag{G.9}\]

Writing down eq. (G.9) with \( \tilde{Z} \) everywhere replaced by \( Z \), and using for \( Z \) the defining eq. (3.14) with \( J \) a general matrix as used above, one finds that eq. (G.9) simply results in eq. (G.1) if one chooses \( k_0' = l' \), \( k_0 = l \) and \( \varepsilon = 0 \). It follows that eq. (G.9) as it stands implies eq. (G.2). This completes the proof.

**Appendix H. Inclusion of shift functions**

The derivation of eq. (7.23) is based on the approximation (2.5), i.e., on the neglect of the real part of \( F_{\mu \nu}(E) \). In this appendix we show that the expression (7.23) for the two-point function does not change its form when the shift functions, i.e., the real parts of \( F_{\mu \nu}(E) \), are included in the derivation. The only modification of the theory arises in the expression (7.7) for the average S-matrix in terms of the \( x_\alpha \) and in the connection (7.8) between the transmission coefficients and the coefficients \( x_\alpha \).

The functions \( F_{\mu \nu}(E) \) are defined in eq. (2.4),

\[ F_{\mu \nu}(E) = \sum_a P \int dE' \frac{W_{\mu a}(E') W_{\nu a}(E')}{E - E'} - i \pi \sum_a W_{\mu a}(E) W_{\nu a}(E), \tag{H.1}\]

where \( P \) stands for the principal value. The principal-value integral can be approximated by factoring the energy-dependence of the matrix elements \( W_{\mu a}(E) \). This energy dependence is largely due to the Coulomb barrier and/or the angular momentum barrier and is accounted for in terms of a penetration
factor. Details of this approximation may be found in chapter 8.7 of ref. [12]; a completely analogous formula is also obtained in the framework of R-matrix theory [14]. The result is

\[ F_{\mu\nu}(E) = \sum_{\alpha} W_{\mu\alpha}(E) W_{\nu\alpha}(E) \{ \Delta_{\alpha} - i\pi \}, \]  

(H.2)

with \( \Delta_{\alpha} \) real. We now incorporate the form (H.2) into the derivation of sections 3 to 7. It is straightforward to see that the definition (3.12b) for the matrix \( W \) now reads

\[ W = \sum_{\alpha} \{ (\pi L_{\alpha\beta} + i\Delta_{\alpha}\delta_{\alpha\beta}) W_{\mu\alpha} W_{\nu\alpha} \}. \]  

(H.3)

Sections 4 to 6 remain unchanged if \( W \) is everywhere taken to be defined by eq. (H.3), and so do eqs. (7.1) to (7.6) and eqs. (7.9) to (7.13). It remains to work out \( S_{aa}, T_a, \) and to use the new formulas for these quantities in the calculations leading from eq. (7.13) to eq. (7.23). Instead of eq. (7.7), we obtain

\[ S_{ab} = \delta_{ab} \frac{1 - i\sigma^0 y^*_\alpha}{1 + i\sigma^0 y_\alpha} \quad \text{with} \quad y_{\alpha} = x_{\alpha}(1 + i\Delta_{\alpha}/\pi). \]  

(H.4)

Equation (7.8) for the transmission coefficient now reads

\[ T_a = 4x_{\alpha} \Delta_0 \left[ (1 + 2\Delta_0 x_{\alpha} + x_{\alpha}^2) + \frac{\Delta_{\alpha} x_{\alpha}}{\pi} \left( \frac{\Delta_{\alpha} x_{\alpha}}{\pi} - \frac{E}{\lambda} \right) \right]^{-1}. \]  

(H.5)

Equations (7.14) change; the change amounts to the replacement of the matrix \( L_{\alpha\beta} \) by the matrix \( L_{\alpha\beta} + i\delta_{\alpha\beta}(\Delta_{\alpha}/\pi) \) on the r.h.s. of each equation. Proceeding as in section 7 but using the results (H.4) and (H.5) instead of eqs. (7.7) and (7.8), we find that eq. (7.23) remains formally unchanged.

**Appendix I. Diagonalisation of \( t_{12} \)**

It is necessary to “diagonalise” \( t_{12} \) explicitly since the angles of the transformation, and the “eigenvalues” of \( t_{12} \), are introduced in section 8 as independent variables of integration. We use quotation marks to indicate that the “diagonalisation” of \( t_{12} \) in eq. (I.18) is accomplished by \( \text{two} \) transformation matrices rather than one. We also emphasize that the new commuting (anticommuting) variables of integration contain an even (odd) polynomial in the original Grassmann variables.

The matrices \( t_{12} \) and \( t_{21} \) are explicitly given in tables D.3. We also recall the symmetry property (D.17) which we write in the form

\[ T_0 = (CKL)T_{0*}(CKL)^{-1}. \]  

(I.1)

We have used eq. (D.12) and suppressed the index c. We specialize eq. (I.1) to \( t_{12} \) and \( t_{21} \), see eq. (5.29). We define the graded 4 × 4 matrices \( C_1 \) and \( C_2 \) by block construction: \( C_1 \) (\( C_2 \)) consists of the 2 × 2 unit matrix and the matrix \( \gamma \) (\( \gamma^T \), respectively) defined in eq. (D.5). Equation (I.1) then reads
\[ t_{12} = C_1 t_{12}^* C_1^T ; \quad t_{21} = C_2 t_{21}^* C_2^T. \]  

We also observe that eqs. (D.15) imply that (see eq. (D.16))

\[ t_{12}^i = k t_{21}. \]  

The "diagonalisation" of \( t_{12} \) (and that of \( t_{21} \)) is accomplished by diagonalising first the graded 4 \( \times \) 4 matrices \( \alpha_p, p = 1, 2 \) defined in eqs. (7.20). From eqs. (I.2) and (I.3) we see that

\[ \alpha_1^* = \alpha_1, \quad \alpha_2^* = k \alpha_2 k, \quad C_p \alpha_p^* C_p^T = \alpha_p \quad \text{for} \quad p = 1, 2. \]  

Writing the \( \alpha_p \) as blocks of 2 \( \times \) 2 matrices and using eqs. (I.4), we have

\[ \alpha_p = \begin{pmatrix} b_p & i^{p-1} \kappa_p^* \\ i^{p-1} \kappa_p & f_p \cdot 1 \end{pmatrix} = D_p + i^{p-1} N_p. \]  

The \( b_p \) are 2 \( \times \) 2 matrices with commuting elements, the \( f_p \) are commuting scalars, and the \( \kappa_p \) are 2 \( \times \) 2 matrices with anticommuting elements. These quantities obey the relations

\[ b_p^* = b_p = b_p^*; \quad f_p^* = f_p; \quad \gamma \kappa_p^* = \kappa_p. \]  

In the last relation (I.5), the matrices \( D_p \) and \( N_p \) are Hermitean; \( D_p \) consists of \( b_p \) and \( f_p \cdot 1 \) and \( N_p \) of \( \kappa_p \) and \( \kappa_p^* \).

We diagonalise the \( \alpha_p \) in two steps. We first remove the anticommuting elements contained in \( N_p \) by the transformation

\[ \alpha_p' = v_p \alpha_p v_p^{-1}; \quad v_p = \exp\{i^{p-1} X_p\}. \]  

We require that the \( X_p \) be odd in the Grassmann variables which are the elements of \( \kappa_p \). We determine the \( X_p \) from the condition that the Fermion–Boson and Boson–Fermion blocks of \( \alpha_p' \) vanish identically. Equation (I.9) below shows that both these requirements can be met. We expand the \( v_p \) in power series with respect to the \( X_p \). These series terminate after the fourth-order terms since each \( \kappa_p \) contains only 4 independent elements. Decomposing \( X_p \)

\[ X_p = X_{p1} + X_{p3}, \]  

into a term \((X_{p1})\) linear and another \((X_{p3})\) of third order in the elements of \( \kappa_p \), we find the conditions

\[ [D_p, X_{p1}] = N_p, \]  

\[ [D_p, X_{p3}] = \frac{1}{3} (-)^{p-1} (X_{p1}^2 N_p - 2 X_{p1} N_p X_{p1} + N_p X_{p1}^2). \]  

Equations (I.9) can be solved consecutively for \( X_{p1} \) and \( X_{p3} \) unless one eigenvalue of \( b_p \) coincides with \( f_p \). These singular points of the transformation (I.7) are dealt with in section 8; we return to them at the end of this appendix. The Hermitecity of \( D_p \) and \( N_p \) implies that \( X_{p1} \) and \( X_{p3} \) are anti-Hermitean,
and the reality properties (I.6) used in eqs. (I.9) show that the $X_p$ have the form

$$X_p = \begin{pmatrix} 0 & -\xi_p^* \\ \xi_p & 0 \end{pmatrix} \quad \text{with} \quad \gamma \xi_p^* = \xi_p.$$  

(I.10)

For the transformation matrices, this implies

$$C_p v_p^* C_p^{-1} = v_p, \quad k^{-1} v_p^* k p^{-1} = v_p^{-1}.$$  

(I.11)

Equations (I.11) show that the $\alpha_i$ of eq. (I.7) also have the properties (I.4) so that

$$\alpha'_i = \begin{pmatrix} b'_i & 0 \\ 0 & f'_{i-1} \end{pmatrix}$$  

with $(b'_i)^T = b'_i = (b'_i)^T$ and $(f'_{i-1})^* = f'_{i-1}$. The real and symmetric $2 \times 2$ matrices $b'_i$ can now be diagonalised by real orthogonal $2 \times 2$ matrices $O_p(2)$. The eigenvalues are denoted by $b'_{i \nu}$, $i = 1, 2$ and are real. Using block construction, we define the graded $4 \times 4$ matrices $O_p$ by $\{O_p(2), 1\}$ and have with

$$u_p = O_p v_p$$  

(I.13)

that

$$u_p \alpha_p u_p^{-1} = \begin{pmatrix} b'_{11} & b'_{12} \\ 0 & f'_{12} \end{pmatrix}$$  

(I.14)

The matrices $u_p$ also obey the conditions (I.11).

The matrices $\alpha_1$ and $\alpha_2$ have the same eigenvalues. This follows from the fact that the eigenvalues of $\alpha_p$ are the zeroes and poles in $x$ of the function $\det g(\alpha_p - x 1)$, from the identity (A.18), and from the cyclic invariance of the graded trace. We accordingly arrange the first two rows and columns of the $O_p$ in such a way that

$$b'_{11} = b'_{22} = b'_1.$$  

Moreover, $f'_1 = f'_{2} = f'$  

(I.15)

We turn to the matrices $t_{12}$ and $t_{21}$ and consider

$$\tilde{t}_{12} = u_1 t_{12} u_2^{-1} \quad \text{and} \quad \tilde{t}_{21} = u_2 t_{21} u_1^{-1}.$$  

(I.16)

These two matrices commute since $2 \tilde{t}_{12} \tilde{t}_{21}$ and $2 \tilde{t}_{21} \tilde{t}_{12}$ are both equal to the r.h.s. of eq. (I.14), see eqs. (I.15). Therefore, $\tilde{t}_{12}$ and $\tilde{t}_{21}$ can be diagonalised simultaneously. Let $u$ be chosen such that $u t_{12} u^{-1}$ and $u t_{21} u^{-1}$ are both diagonal. Since $\tilde{t}_{12} \tilde{t}_{21}$ and $\tilde{t}_{21} \tilde{t}_{12}$ are diagonal already, the matrix $u$ can differ from the unit matrix only in the lower right-hand $2 \times 2$ block since there the eigenvalues of the $\alpha_p$ are degenerate, see eq. (I.14). (We do not pay attention to possible accidental degeneracies of the $b'_i$ or of the $b'_1$ and $f'$.) This shows that $t_{12}$ and $t_{21}$ each are diagonal except for this same lower right-hand $2 \times 2$ block. The symmetry properties of $\tilde{t}_{12}$ and $\tilde{t}_{21}$ are the same as those of $t_{12}$ and $t_{21}$, eqs. (I.2) and (I.3), since the transformation matrices $u_p$ obey eqs. (I.11). It thus follows from tables D.3 and eqs. (5.21) that $\tilde{t}_{12}$ and
\[ \tilde{t}_{12} = \begin{pmatrix} (\mu_1 & \emptyset \\ \emptyset & \mu_2 \end{pmatrix} i\mu U \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}, \quad \tilde{t}_{21} = \begin{pmatrix} (\mu_1 & \emptyset \\ \emptyset & \mu_2 \end{pmatrix} i\mu U^+ \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}. \quad (1.17) \]

Here, the "eigenvalues" \( \mu_1, \mu_2 \) and \( \mu \) are real, \( \text{ord}(\mu) \geq 0 \), and \( U \) is special unitary. It is convenient to use eqs. (1.17) rather than a complete diagonalisation of \( t_{12} \) and \( t_{21} \). Collecting everything, we have

\[ t_{12} = u_1^{-1} U \begin{pmatrix} \mu_1 & \mu_2 \\ \emptyset & i\mu \end{pmatrix} u_2, \quad t_{21} = u_2^{-1} \begin{pmatrix} \mu_1 & \mu_2 \\ \emptyset & i\mu \end{pmatrix} U^+ u_1. \quad (1.18) \]

We use a notation which does not distinguish between the \( 2 \times 2 \) matrix \( U \) in eq. (1.17), and the \( 4 \times 4 \) matrix given in block construction by \( (1, U) \). The permissible range of values of \( \mu_1, \mu_2 \) and \( \mu \) can be inferred from eqs. (1.1) and (5.29) which imply that \( (1 + \tilde{t}_{12} \tilde{t}_{21})^{1/2} \) and \( (1 + \tilde{t}_{21} \tilde{t}_{12})^{1/2} \) be real. These conditions are always met by \( \mu_1 \) and \( \mu_2 \). For \( \mu \) we find, using also the condition \( \text{ord}(\mu) \geq 0 \), that

\[ 0 \leq \text{ord}(\mu) \leq 1. \quad (1.19) \]

Here, the symbol \( \text{ord} \) refers to the ordinary part of \( \mu \) (the term of zeroth order in the anticommuting variables). The condition (1.19) is the quantitative formulation of the statement that with respect to the Fermion–Fermion block, the saddle-point manifold is compact.

According to eq. (1.18), \( t_{12} \) is now parametrized in terms of the following variables: Three commuting variables \( \mu_1, \mu_2, \mu \), three commuting variables defining the special unitary transformation \( U \) (see appendix K), two commuting variables, each defining one of the two orthogonal transformations \( O_1 \) and \( O_2 \), and 8 anticommuting variables \( (4 \text{ contained on both } \kappa_1 \text{ and } \kappa_2) \). The total number of variables – 8 commuting and 8 anticommuting ones – is, of course, the same as in the parametrization of table D.3.

Returning to the diagonalisation of the matrices \( \alpha_p \) in eqs. (1.14) and (1.15), we notice that eqs. (7.20) and (1.18) imply \( b_1^i = 2\mu_1^2, \quad b_2^i = 2\mu_2^2 \) and \( f' = -2\mu^2 \). The solutions of eqs. (1.9) are singular whenever \( \text{ord}(b_1^i) = \text{ord}(f') \) for \( i = 1 \) or 2. We see now that this happens only when the ordinary parts of \( \mu_1, \mu_2 \) and \( \mu \) obey the conditions

\[ \text{ord}(\mu_1) = 0 = \text{ord}(\mu) \quad \text{or} \quad \text{ord}(\mu_2) = 0 = \text{ord}(\mu) \quad \text{or both}. \quad (1.20) \]

In eq. (5.36) it was assumed that the matrix \( A \) can be diagonalised by a matrix \( T \) obeying the symmetry relations (D.12), (D.17). Following the construction which leads to eqs. (1.18) we now show that this is indeed the case. We recall the definition (4.3) of \( A \) (this definition includes the substitution \( \psi \rightarrow \hat{\psi} \)), and consider instead of \( A \) the matrix

\[ \hat{A}_{\alpha\beta} = [(KL)^{1/2}]_{\alpha\gamma} \hat{\psi}_{\mu\gamma} \hat{\psi}_{\mu\delta}^\dagger [(KL)^{1/2}]_{\delta\beta}. \quad (1.21) \]

The matrix \( \hat{A} \) differs from \( A \) by a factor \( i\lambda \) (which is omitted for convenience). Equations (1.21) and (E.3) imply

\[ (CL)\hat{A}^\ast (C^T L) = \hat{A}, \quad (KL)\hat{A}^\dagger (KL) = \hat{A}. \quad (1.22) \]
The diagonalisation of $\hat{A}$ proceeds in two steps. As in the development leading from eqs. (1.5) to eqs. (1.11), we first transform $\hat{A}$ by a pseudo-unitary transformation $V$,

$$\hat{A}' = V\hat{A}V^{-1},$$

(1.23)

in such a way that $\hat{A}'$ has vanishing elements on the Fermion–Boson, and on the Boson–Fermion blocks. Explicit calculation shows that this is accomplished by a pseudo-unitary matrix $V$ which obeys

$$(CL)V^*(C^TL) = V, \quad V^\dagger KV = KL.$$ (1.24)

Using eq. (E.4) we see that the transformation $V$ induces on $\hat{\psi}$ a transformation $\hat{V}$ which obeys eqs. (D.17). Hence, $\hat{V}$ is an element of the group of transformations $T_c$, and therefore $V$ is an element of the group of transformations $\hat{T}$ introduced in eq. (E.7). The singularities connected with eqs. (1.9) do not arise in the present context, for the following reason. A singularity could arise only if an eigenvalue of the Boson–Boson block of $\hat{A}$ vanishes. The structure of the matrix $\hat{A}$ (see table 4.1) shows that this happens only when the $N$ four-vectors $\{S_\mu(p)\}, \mu = 1, \ldots, N$ span, at most, a three-dimensional space. In that case, however, the rows of the submatrix $\hat{A}_{\alpha\beta}$ with $\alpha \leq 4$, $\beta \geq 5$, and the columns of the submatrix $\hat{A}_{\alpha\beta}$ with $\alpha \geq 5$, $\beta \leq 4$ are linearly dependent. In other words, if an eigenvalue of the Boson–Boson block of $\hat{A}$ vanishes we can find a linear combination of the first four rows and columns of $\hat{A}$ such that the first row and column of the reordered matrix vanish identically. The diagonalisation now applies only to the submatrix consisting of rows and columns labelled two or higher, and a singularity therefore does not arise.

The Fermion–Fermion block of the matrix $\hat{A}'$ (which also obeys eqs. (1.22)) has the same structure as that of the matrix $\sigma$ in table 4.2. Proceeding as in section 5.3, we can extract a special unitary matrix, and diagonalise the remainder in terms of an orthogonal transformation. Both the unitary matrix and the orthogonal transformation obey the symmetry relations (1.24) and are therefore members of the group of transformations $T_c$. The diagonalisation of the Boson–Boson part of $\hat{A}'$ can finally be accomplished in a manner analogous to the one described in appendix A of ref. [2]. The diagonalising matrix again obeys eqs. (1.24). This completes the proof.

Appendix K. Invariant measure and calculation of the volume element $d\mu(t)$

In this appendix, we evaluate $d\mu(t)$. We do so without paying attention to overall sign factors. In contrast to the case of commuting variables (where the Jacobian is the modulus of the determinant of the partial derivatives of the old variables with respect to the new ones) sign factors are important in the case of anticommuting variables since the relative order of the differentials of these variables is not arbitrary. We determine the overall sign at the end by the requirement that $|S_{ab}|^2 - |S_{ab}|^2$ be positive.

The differential $d\mu(t)$ is defined in eq. (F.6). We also recall the definition of $\delta T_{12}$ in appendix F and write $d\mu(t)$ as

$$d\mu(t) = |\text{det}_g[(\delta T_0)^{-1}T^{-1}_{12}/\delta t_{12}]|d[t_{12}],$$ (K.1)

with $T_0$ given by eq. (5.29). It is convenient to decompose the evaluation of $d\mu(t)$ into several steps. This is done by introducing graded $4 \times 4$ matrices $\tau_{12}$ and $\tau_{21}$ which are defined implicitly by
The matrix \( T_0 \), given in eq. (5.29), takes the form
\[
T_0 = \begin{pmatrix}
(1 + \tau_{12} \tau_{21}) (1 - \tau_{12} \tau_{21})^{-1} & 2i(1 - \tau_{12} \tau_{21})^{-1} \\
-2i(1 - \tau_{12} \tau_{21})^{-1} \tau_{21} & (1 + \tau_{21} \tau_{12}) (1 - \tau_{21} \tau_{12})^{-1}
\end{pmatrix}.
\] (K.3)

This "rational" parametrization is useful for the calculation of certain derivatives. We write eq. (K.1) as
\[
d\mu(t) = |\text{det}(\delta T_0 T_0^{-1})_{12}/\delta \tau_{12}| \text{det}(\delta \tau_{12}/\delta t_{12}) |d[t_{12}]|
\] (K.4)
and we evaluate the factors separately. Several steps of the calculation relate to the invariant measures of subgroups or coset spaces of the group UOSP (2,2/4). We do not utilize this fact and carry out the calculation explicitly, mentioning these connections only in passing.

### K.1. Calculation of \( \text{det}(\delta T_0 T_0^{-1})_{12}/\delta \tau_{12} \)

We use [1, 2] block notation and the fact that \( T_0^{-1} = L T_0 L \). Then,
\[
(\delta T_0 T_0^{-1})_{12} = (\delta T_0)_{12} (T_0)_{22} - (\delta T_0)_{11} (T_0)_{12}.
\] (K.5)

With the parametrization (K.3), this becomes
\[
(\delta T_0 T_0^{-1})_{12} = 2i(1 - \tau_{12} \tau_{21})^{-1} (\delta \tau_{12} - \tau_{12} \delta \tau_{21} \tau_{12}) (1 - \tau_{21} \tau_{12})^{-1}.
\] (K.6)

We observe that the "diagonalisation" of \( t_{12} \) and \( t_{21} \) in eqs. (1.18) also diagonalises \( \tau_{12} \) and \( \tau_{21} \) (modulo the special unitary matrix \( U \)). We denote the common "eigenvalues" of these two latter matrices by \( (\theta_1, \theta_2, i\theta, i\theta) \), and the diagonal \( 4 \times 4 \) matrix containing the "eigenvalues" by \( \vartheta \). Then,
\[
[(\delta T_0) T_0^{-1}]_{12} = 2i u_1^{-1} (1 - \vartheta^2)^{-1} u_1 \delta \tau_{12} u_2^{-1} (1 - \vartheta^2)^{-1} u_2 \\
- 2i u_1^{-1} U(1 - \vartheta^2)^{-1} \vartheta u_2 \delta \tau_{21} u_1^{-1} U(1 - \vartheta^2)^{-1} \vartheta u_2.
\] (K.7)

We write this in the suggestive fashion
\[
[u_1 [(\delta T_0) T_0^{-1} u_2^{-1} U^\tau] = 2i (1 - \vartheta^2)^{-1} [u_1 (\delta \tau_{12}) u_2^{-1} U^\tau] (1 - \vartheta^2)^{-1} \\
- 2i (1 - \vartheta^2)^{-1} \vartheta [U u_2 (\delta \tau_{21}) u_1^{-1}] (1 - \vartheta^2)^{-1} \vartheta.
\] (K.8)

We note that
\[
[u_1 \delta \tau_{12} u_2^{-1} U^\tau] = k [U u_2 \delta \tau_{21} u_1^{-1}].
\] (K.9)

Equation (K.9) shows that \( u_1 \delta \tau_{12} u_2^{-1} U^\tau \) and \( U u_2 \delta \tau_{21} u_1^{-1} \) are related in the same way as \( t_{12} \) and \( t_{21} \), cf. eq. (I.3). Moreover, we have from eqs. (I.11) and (5.21) that
\( (u_1 \delta \tau_{12} u_2^{-1} U') = C_1 (u_1 \delta \tau_{12} u_2^{-1} U')^* C_2 \)  

(K.10)

which shows that \((u_1 \delta \tau_{12} u_2^{-1} U')\) obeys the same symmetry relation as \(t_{12}\) in eq. (I.2). All this shows that the elements of \((u_1 \delta \tau_{12} u_2^{-1} U')\) can be arranged in the form of the first of tables D.3, while the elements of \((U u_2 \delta \tau_{31} u_1^{-1})\) then take the form of the second of tables D.3.

The matrix \(((\delta T_0) T_0^{-1})_{12}\) has 16 elements which we denote by \(\delta x_i, i = 1, \ldots, 16.\) Likewise, the 16 elements of \(\delta \tau_{12}\) are denoted by \(\delta y_i, i = 1, \ldots, 16.\) The 16 elements of the matrix \((u_1 (\delta T_0) T_0^{-1}) u_2^{-1} U')\) are denoted by \(\Sigma_{i=1}^{16} A_{ki} \delta x_i\) with \(k = 1, \ldots, 16.\) The 16 elements of the matrix \((u_1 \delta \tau_{12} u_2^{-1} U')\) are of the form \(\Sigma_{i=1}^{16} A_{ki} \delta y_i, k = 1, \ldots, 16,\) with the same coefficients \(A_{ki}.\) The identity

\[
\detg \left[ \sum_i A_{ki} \delta x_i / \sum_j A_{kj} \delta y_j \right] = \detg[\delta x_i / \delta y_i] 
\]

(K.11)

shows that we may take the elements of \((u_1 (\delta T_0) T_0^{-1})_{12}\) and of \((u_1 \delta \tau_{12} u_2^{-1} U')\) [rather than the elements of \((\delta T_0) T_0^{-1})_{12}\) and of \(\delta \tau_{12}\) as the independent variables in the evaluation of \(\detg[\delta \tau_{12}].\) This fact, the diagonality of \(\delta \tau,\) and the use of tables D.3 greatly simplify the calculation. The graded determinant decays into a product. Each factor is the graded determinant of a matrix of dimension \(\leq 2.\) We obtain altogether

\[
|\detg[\delta T_0 T_0^{-1})_{12}/\delta \tau_{12}]| = \frac{(1 - \theta_i^2 \theta_j^2) (1 - \theta^2)}{(1 + \theta_i^2 \theta_j^2) (1 + \theta_i^2 \theta_j^2)} \cdot \frac{(1 - \theta_i^2) (1 - \theta_j^2)}{(1 + \theta_i^2) (1 + \theta_j^2)}.
\]

(K.12)

K.2. Calculation of \(d[\tau_{12}]\) and of \(d[t_{12}]\)

The transformation (I.18) “diagonalises” both \(t_{12}\) and \(t_{21}\) and, because of eqs. (K.2), also \(\tau_{12}\) and \(\tau_{21}.\) We express \(d[\tau_{12}]\) and \(d[t_{12}]\) in terms of the differentials of the new variables, and of the Jacobian of the transformation (I.18). We do the calculation for \(d[t_{12}]\) and simply write down the analogous result for \(d[\tau_{12}].\)

To calculate \(d[t_{12}]\) we vary eq. (I.18) and obtain in analogy to eq. (F.2)

\[
\delta t_{12} = u_1^{-1} U [-\delta O_i \mu_o - \delta v_i \mu_o + \delta U' \mu_o + \delta \mu_o + \mu_o \delta O_+ + \mu_0 \delta v'o] u_2.
\]

(K.13)

Here, \(\mu_0\) is a 4 \(\times\) 4 matrix containing the “eigenvalues” \((\mu_1, \mu_2, i\mu, i\mu)\) as elements, and the primed matrices are defined by

\[
\delta O'_i = U^* (\delta O_i) O'_i U; \quad \delta O'_2 = (\delta O_2) O'_2; \quad \delta U' = U^* \delta U;
\]

\[
\delta v'_i = (U^* O_i) (\delta v_i v'_i^{-1}) (O'_i U); \quad \delta v'_2 = O_2 (\delta v_2 v'_2^{-1}) O'_2.
\]

(K.14)

In the evaluation of the graded determinant of the 16 \(\times\) 16 partial derivatives, the matrices \(u_1^{-1}, U\) and \(u_2\) multiplying the bracket on the r.h.s. of eq. (K.13) appear in the form of powers of \(\detg u_1, \detg U\) and \(\detg u_2.\) Each of these factors is individually equal to unity. For \(U,\) this follows since \(U\) is special unitary. For the \(u_p\) with \(p = 1, 2\) it follows from \(u_p = O_p v_p,\) from the fact that the \(O_p\) are orthogonal, and from the fact that \(v_p^{-1} = k v_p k.\) The last statement is a direct consequence of the form of \(v_p\) as given in eqs. (I.7) and (I.10). We therefore focus attention on the content of the bracket in eq. (K.13) and
proceed as in appendix F. We evaluate the Jacobian in steps, by first using the elements of \( \delta O_p \), \( \delta U' \) etc. as independent variables, and by calculating subsequently the Jacobians \( \mathcal{F}(\delta O_p/\delta O_p) \), \( \mathcal{F}(\delta U'/\delta U) \) etc.

The 2×2 matrices \( O_p(2) \) each depend on a single variable \( \varphi_p \). Using the standard representation, we find

\[
\delta O_p = -\gamma \delta \varphi_p
\]

(K.15)

with \( \gamma \) defined in eq. (D.5). We notice that the r.h.s. of eq. (K.15) coincides with \( \delta O_p \) so that \( \mathcal{F}(\delta O_p/\delta O_p) = 1 \). (This shows that the invariant measure of O(2) is unity).

It is easily shown that \( \delta U' \) has the same symmetry properties as the generators of the group SU(2). We therefore parametrize \( \delta U' \) in the same way as was done for the 2×2 matrix in the lower right-hand block of table D.1. We denote the elements by ±i \( dm' \), \( -dm_1' \) and \( +dm_1'' \), respectively. The primes indicate that \( \delta U' \) is not evaluated near the group unit \( U = 1 \), but anywhere in the group SU(2).

Using the symmetry relation (I.11) we show that the \( \delta U' \) have the same symmetry properties as the generators of the matrices \( U_p \). In particular, the 2×2 matrices in the upper right-hand and the lower left-hand blocks have the same form as in eq. (I.10). We accordingly parametrize the 2×2 matrices in the lower left-hand block as

\[
j_p^{-1} \begin{pmatrix} \delta \alpha_p' & \delta \beta_p' \\ \delta \alpha_p'' & \delta \beta_p'' \end{pmatrix}, \quad p = 1, 2.
\]

(K.16)

The primes have the same significance as in the parametrization of \( \delta U' \). The matrices \( \delta U' \) have non-vanishing terms also in the upper left-hand and the lower right-hand blocks; it is shown below that these contributions do not affect the value of the Jacobian.

Using the parametrization of \( \delta O_p' \), of \( \delta U' \), and of \( \delta U_p' \) just introduced, we find that the content of the bracket in eq. (K.13), written as a 4×4 matrix, takes the form of table K.1. The dots in the 2×2 matrices in the upper left-hand and the lower right-hand blocks indicate additional terms due to the \( \delta U_p' \) which are not given explicitly. The dependence of the various elements in \( \delta t_{12} \) on \( \delta \mu_0 \), \( \delta O_p ' \) etc. is schematically indicated in table K.2 which is completely analogous to table F.1. The symbols \( (\delta t_{12})_{x'x} \) (with \( x \) and \( x' \) equal to B or F) denote the various 2×2 matrices in \( \delta t_{12} \). The zeroes in the various blocks

<table>
<thead>
<tr>
<th>Table K.1</th>
<th>Differentials contributing to eq. (K.13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta \mu_1 + \cdots )</td>
<td>( -\mu_2 \delta \varphi_1 + \mu_1 \delta \varphi_2 + \cdots )</td>
</tr>
<tr>
<td>( \mu_1 \delta \varphi_1 - \mu_2 \delta \varphi_2 + \cdots )</td>
<td>( \delta \mu_2 + \cdots )</td>
</tr>
<tr>
<td>( -\mu_1 \delta \alpha_1 )</td>
<td>( -\mu_2 \delta \alpha_1 )</td>
</tr>
<tr>
<td>( -\mu_1 \delta \alpha_1 - \mu \delta \alpha_2 )</td>
<td>( -\mu_2 \delta \alpha_1 - \mu \delta \alpha_2 )</td>
</tr>
<tr>
<td>( -\mu_1 \delta \alpha_1 + \mu \delta \alpha_2 )</td>
<td>( -\mu_2 \delta \alpha_1 + \mu \delta \alpha_2 )</td>
</tr>
<tr>
<td>( -\mu_1 \delta \beta_1 - \mu \delta \beta_2 )</td>
<td>( -\mu_2 \delta \beta_1 - \mu \delta \beta_2 )</td>
</tr>
<tr>
<td>( -\mu_1 \delta \beta_1 + \mu \delta \beta_2 )</td>
<td>( -\mu_2 \delta \beta_1 + \mu \delta \beta_2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table K.2</th>
<th>Dependence of ( \delta t_{12} ) on ( \delta \mu_0 ), ( \delta O_p ' ), ( \delta U' ) and ( \delta U_p ' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\delta t_{12})_{HB} )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( (\delta t_{12})_{HF} )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( (\delta t_{12})_{BF} )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>
in Table K.2 show immediately that the Jacobian factorises, and that the knowledge of the dotted terms in Table K.1 is not needed. The calculation of the Jacobian with the help of Table D.3 is straightforward; the absolute value is given by

\[ 2 \mu^3 \frac{|\mu_1^2 - \mu_2^2|}{(\mu_1^2 + \mu_2^2)^2} \]  

(K.17)

Of the remaining Jacobians \( \mathcal{F}(\delta U'/\delta U) \) and \( \mathcal{F}(\delta v_p/\delta v_p) \), the first one is proportional to the invariant measure of SU(2), and the second one is proportional to the invariant measure associated with the coset space UOSP(2/2)/O(2) \( \otimes \) SU(2). [The matrices \( v_p \) do not close under group multiplication as is for example seen from \( (\delta v_p v_p^{-1})_{xx} \neq 0 \).]

To work out \( \mathcal{F}(\delta U'/\delta U) \), we parametrize the matrices \( U \) in the form

\[ U = (1 + m^2 + r^2 + s^2)^{-1/2} \begin{pmatrix} 1 + im & -(r + is) \\ r - is & 1 - im \end{pmatrix} \]  

(K.18)

with \( m, r, s \) real and \( -\infty < m, r, s < +\infty \). The matrices (K.18) are special unitary. For \( r, s, m \) infinitesimal and \( m_1 = r + is \), eq. (K.18) is obviously consistent with Table D.1. We calculate

\[ \delta U' = U'(m, r, s) \left[ U(m + \delta m, r + \delta r, s + \delta s) - U(m, r, s) \right] \]  

(K.19)

to first order in \( \delta m, \delta r \) and \( \delta s \) and compare the result with the parametrization of \( \delta U' \) in terms of \( dm' \), \( dm'_1 \), \( dm'_1^* \) used in Table K.1. We find

\[ \delta m' = p \delta m + ps \delta r - pr \delta s, \]

\[ \delta m'_1 = ip (r + is) \delta m + p (1 + im) \delta r + ip (1 - im) \delta s, \]  

(K.20)

\[ \delta m'_1^* = -ip (r - is) \delta m + p (1 + im) \delta r - ip (1 - im) \delta s, \]

where \( p = (1 + m^2 + r^2 + s^2)^{-1} \). This yields

\[ |\mathcal{F}(\delta U'/\delta U)| = 2 \left[ 1 + m^2 + r^2 + s^2 \right]^{-2} \]  

(K.21)

with an associated product of differentials given by \( dm \, dr \, ds \). The r.h.s. of eq. (K.21) gives the invariant measure of SU(2) for the parametrization (K.18).

To evaluate \( \mathcal{F}(\delta v_p/\delta v_p) \) we observe that the matrices \( (U^\dagger O_1) \) and \( O_2 \) in the definitions (K.14) of \( \delta v'_1 \) and \( \delta v'_2 \) yield factors \( \det U^\dagger = 1 \), \( \det O_1 = 1 \) and \( \det O_2 = 1 \). We therefore consider only \( \mathcal{F}(\delta v_p v_p^{-1})/\delta v_p \). As in the case of \( \mathcal{F}(\delta U'/\delta U) \), we determine the Jacobian from a comparison between the parametrization of \( \delta v'_p \) as used in formula (K.16) and Table K.1, and the direct calculation of the lower left-hand block of \( \delta v_p v_p^{-1} \) in terms of the parametrization of \( v_p \) given in Appendix I. We recall the second of Eqs. (I.7) and eqs. (I.10). We write the \( 2 \times 2 \) matrices \( \xi_p \) in the form

\[ \xi_p = \begin{pmatrix} \hat{a}_p^{\dagger} & \hat{b}_p \\ \hat{a}_p & \hat{b}_p^{\dagger} \end{pmatrix}. \]  

(K.22)
The anticommuting variables $\alpha_p$, $\beta_p$ with $p = 1, 2$ are determined by the solutions of eqs. (1.9). We recall eq. (I.8) and the fact that the $\alpha_p$, $\beta_p$ contain terms of first and third order in the anticommuting variables $\eta_p$, $\rho_p$ etc. occurring in table D.3. The calculation of the Jacobian $\mathcal{F}(\delta v_p, v_p^{-1}/\delta v_p)$ greatly simplifies if we use instead of the variables $\alpha_p$, $\beta_p$, $\alpha_p^*$, $\beta_p^*$ introduced in eq. (K.22) another set $\tilde{\alpha}_p$, $\tilde{\beta}_p$, $\tilde{\alpha}_p^*$, $\tilde{\beta}_p^*$ defined by

\[
\xi_p = \begin{pmatrix} \tilde{\alpha}_p & \tilde{\beta}_p \\ \tilde{\alpha}_p^* & \tilde{\beta}_p^* \end{pmatrix} = (1 - \frac{1}{2}(-)^p \tilde{\alpha}_p) \xi_p
\]

where $\tilde{\alpha}_p = \alpha_p \tilde{\alpha}_p^* + \beta_p \tilde{\beta}_p^*$. We now express $v_p^{z\pm 1}$ in terms of these new variables, and from there calculate $\mathcal{F}(\delta v_p, v_p^{-1}/\delta v_p)$.

Inverting eqs. (K.23) and defining $A_p = \alpha_p \tilde{\alpha}_p^* + \beta_p \tilde{\beta}_p^*$, we have

\[
\xi_p = (1 + \frac{1}{3}(-)^p A_p) \xi_p ; \quad p = 1, 2.
\]

We define

\[
Y_p = \begin{pmatrix} \theta & -\xi_p^* \\ \xi_p & \theta \end{pmatrix}.
\]

It is easy to check that $\xi_p^{\top} \xi_p = A_p \cdot 1$, and that therefore $Y_p^2 = -A_p Y_p$. Since eqs. (K.24) and (I.10) imply $X_p = (1 + \frac{1}{3}(-)^p A_p) Y_p$, the last relation yields $X_p = Y_p - \frac{1}{3}(-)^p Y_p^3$. We use this in eq. (I.7) and find for $v_p^{z\pm 1}$ the expansion

\[
v_p^{z\pm 1} = 1 \pm i^{p-1} Y_p + \frac{1}{2} i^{2(p-1)} Y_p^2 \pm \frac{1}{3} i^{3(p-1)} Y_p^3 + \frac{1}{4} Y_p^4.
\]

From eq. (K.26), it is straightforward to calculate $\delta v_p$. In evaluating $\delta v_p v_p^{-1}$ we need to keep only terms with even powers of $Y_p$ since – owing to the structure of $Y_p$ in eq. (K.25) – only these terms contribute to the $2 \times 2$ matrix in the lower left-hand block which we eventually compare with expression (K.16). We find

\[
(\delta v_p, v_p^{-1})_{\text{even}} = i^{p-1}(\delta Y_p + \frac{1}{2}(-)^p (Y_p^2 \delta Y_p + \delta Y_p Y_p^2) - \frac{1}{3} Y_p^3 \delta Y_p Y_p + \frac{1}{4} Y_p^4 Y_p Y_p + \frac{1}{5} Y_p \delta Y_p Y_p Y_p^3)
\]

Using the explicit form (K.25) and identities like $\xi_p^{\top} \xi_p = A_p \cdot 1$, we find from eq. (K.27) for the above-mentioned $2 \times 2$ matrix the expression

\[
(\delta v_p, v_p^{-1})_{\text{even}} = i^{p-1}(1 + \frac{1}{2}(-)^p A_p - \frac{1}{3} A_p^2) \delta \xi_p + \frac{1}{2}(-)^p \delta \xi_p (\xi_p^{\top} \xi_p) - \frac{1}{3} (-)^p (\delta \alpha_p \beta_p + \alpha_p \delta \beta_p) \delta \beta_p + \frac{1}{2}(-)^p (\delta \alpha_p \beta_p^* + \alpha_p \delta \beta_p^*) \delta \beta_p^*.
\]

Inserting the explicit form for $\xi_p$ from eq. (K.23) and comparing with eq. (K.16), we obtain

\[
\delta \alpha_p = (1 + \frac{1}{2}(-)^p (\beta_p \beta_p^* - \tilde{\alpha}_p \tilde{\alpha}_p^*) - \frac{1}{3} \alpha_p \tilde{\alpha}_p^* \beta_p \tilde{\beta}_p^*) \delta \tilde{\alpha}_p - \frac{1}{2}(-)^p (\alpha_p \delta \beta_p - \beta_p \delta \alpha_p) \delta \beta_p,
\]

\[
\delta \beta_p = (1 + \frac{1}{2}(-)^p (\alpha_p \tilde{\alpha}_p^* - \pi_p \pi_p^*) - \frac{1}{3} \alpha_p \tilde{\alpha}_p^* \beta_p \tilde{\beta}_p^*) \delta \beta_p - \frac{1}{2}(-)^p (\alpha_p \delta \beta_p + \beta_p \delta \alpha_p) \delta \alpha_p.
\]
The equations for $\delta \alpha_{p}^{*}$ and $\delta \beta_{p}^{*}$ follow by complex conjugation. The Jacobian is a product of two determinants of two $2 \times 2$ matrices. This simplification is due to the transformation (K.23). Each determinant is unity.

Collecting everything, we have
\[
d[t_{12}] = 4[1 + m^2 + r^2 + s^2]^{-2} \mu^3 \frac{|\mu_1^2 - \mu_2^2|}{(\mu_1^2 + \mu_2^2)^2 (\mu_1^2 + \mu_2^2)^2} 
\cdot \left( \prod_{p=1}^{2} d\alpha_p^* d\beta_p^* d\beta_p^* d\alpha_p^* \right) d\varphi_1 d\varphi_2 dm dr ds d\mu_1 d\mu_2.
\]

(K.30)

Analogously, we have
\[
d[\tau_{12}] = 4[1 + m^2 + r^2 + s^2]^{-2} \theta^3 \frac{|\theta_1^2 - \theta_2^2|}{(\theta_1^2 + \theta_2^2)^2 (\theta_1^2 + \theta_2^2)^2} 
\cdot \left( \prod_{p=1}^{2} d\alpha_p^* d\beta_p^* d\beta_p^* d\alpha_p^* \right) d\varphi_1 d\varphi_2 dm dr ds d\theta d\theta_1 d\theta_2.
\]

(K.31)

K.3. Evaluation of $d\mu(t)$

Combining eqs. (K.12) and (K.31), we have
\[
det[(\delta T_0^{-1})_{12}/\delta \tau_{12}] d[\tau_{12}] = 4[1 + m^2 + r^2 + s^2]^{-2}
\cdot \frac{|(1 - \theta_1^2 \theta_2^2) (1 - \theta_1^2)|}{(1 + \theta_1^2 \theta_2^2)^2 (1 + \theta_2^2 \theta_2^2)^2}
\cdot \frac{|(1 - \theta_1^2) (1 - \theta_2^2)|}{(1 + \theta_1^2)^2 (1 + \theta_2^2)^2}
\cdot \frac{\theta^3 |\theta_1^2 - \theta_2^2|}{(\theta_1^2 + \theta_2^2)^2 (\theta_1^2 + \theta_2^2)^2}
\cdot \left( \prod_{p=1}^{2} d\alpha_p^* d\beta_p^* d\beta_p^* d\alpha_p^* \right) d\varphi_1 d\varphi_2 dm dr ds d\theta d\theta_1 d\theta_2.
\]

(K.32)

Equations (K.2) imply the relations
\[
\mu_i = 2(1 - \theta_i^2)^{-1} \theta_i, \quad i = 1, 2; \quad \mu = 2(1 + \theta^2)^{-1} \theta.
\]

(K.33)

These relations can be used to calculate $d\theta/d\mu_i; d\theta_j/d\mu_i$, $i = 1, 2$; and to express the factors on the r.h.s. of eq. (K.32) involving $\theta_1, \theta_2$ and $\theta$ in terms of $\mu_1, \mu_2$ and $\mu$. Comparing the result with eq. (K.30), we obtain
\[
d\mu(t) = \frac{(1 - \mu^2)}{(1 + \mu_1^2)^{1/2} (1 + \mu_2^2)^{1/2}} d[t_{12}].
\]

(K.34)

Equations (K.34) and (K.30) constitute the result of this appendix. We note that the explicit calculation of $det[\delta \tau_{12}/\delta t_{12}]$, suggested originally by eq. (K.4), has been simplified by virtue of the route leading to eqs. (K.32) and (K.34).
Appendix L. Derivation of eq. (8.6), and of the substitution rules (8.3)

To guide the reader through the technicalities of this appendix, and to assist him in appreciating the power and simplicity of the result, we begin by explaining the origin of the substitution rules (8.3) in a qualitative fashion.

The basic step of section 8 is to express the pre-exponential factor in terms of the “eigenvalues” \( \mu_1, \mu_2, \mu \), and of the Grassmann variables \( \hat{\alpha}_p, \hat{\alpha}_p^*, \hat{\beta}_p, \hat{\beta}_p^* \) which parametrize the matrices \( v_p \). After substitution of eq. (8.1) into eq. (7.23) each pre-exponential term becomes a sum of terms which we classify according to their order in the Grassmann variables. Only terms of even order occur. Terms which depend only on the eigenvalues \( \mu_1, \mu_2, \mu \) but not on \( \hat{\alpha}_p, \hat{\alpha}_p^*, \hat{\beta}_p, \hat{\beta}_p^* \) (terms of zeroth order) are called invariant. With these definitions, the substitution rules (8.3) amount to the statement that all non-invariant terms with the exception of the term of maximum (eighth) order integrate to zero, and that the invariant terms vanish, too. [Generally speaking, invariant terms may also give rise to non-vanishing integrals, but this is not the case in the context of the rules (8.3).]

Deferring technical details to the discussion below, we here motivate this statement as follows. The terms of second, fourth and sixth order can be generated by “variation” of a corresponding term of higher order. For example, the term \( Q_1 = f(\mu_1, \mu_2, \mu) \hat{\alpha}_1 \hat{\beta}_1 \) is generated from \( Q_2 = f(\mu_1, \mu_2, \mu) \hat{\alpha}_1^* \hat{\alpha}_2^* \) by making the replacement

\[
\hat{\alpha}_1 \rightarrow \hat{\alpha}_1 + \rho, \quad \hat{\alpha}_2 \rightarrow \hat{\alpha}_2^* + \rho^*.
\] (L.1)

In the remainder of this appendix we essentially argue that variations similar to formula (L.1) leave the integral invariant. Denoting the integrals with \( Q_1 \) (\( Q_2 \)) as pre-exponential factors by \( q_1 \) (\( q_2 \)), respectively, we then have that

\[
q_2 = q_2 + q_1 \rho^* \rho
\] (L.2)

which implies \( q_1 = 0 \) as claimed.

These considerations illustrate the special role played by the term of eighth order: it already contains the maximum number of Grassmann variables! In other words, it is the only term that cannot be generated by variation of a term of higher order, and no such relation as eq. (L.2) applies.

Because of the matrix structure of the expressions \( u_p L(p) u_p^{-1} \) it turns out that the term of zeroth order cannot be obtained directly by variation of terms of higher order. Different arguments are used to generally evaluate such invariant terms. The very invariance of these terms can be used to show that their value is given by \( f(0) \). These last statements have their most elegant and concise expression in Wegner’s integral theorem [17].

It appears that the aforementioned result was first discovered empirically by Efetov [11]. However, this author simply replaces the pre-exponential factor by the term of maximum order without explaining or even motivating the procedure.

We proceed as follows. Using arguments of the type leading to eq. (L.2), we first prove that in the substitution rules (8.3) it is justified to omit the terms involving \( Y_p^k \) with \( 1 \leq k \leq 3 \). We then consider integrals over invariant terms. We show that eq. (8.6a) is valid, and we justify the omission of terms of order zero in \( Y_p \) in the substitution rules (8.3).

We recall the parametrization (5.41) of \( \sigma_G \) and the fact that the matrices \( T_0 \) form a coset space. Replacing in eq. (5.41) the matrix \( T_0 \) by the product \( T_{01} \cdot T_{02} \), with both \( T_{01} \) and \( T_{02} \) of the form (5.29), is
a legitimate operation and leads again to a matrix \( \sigma' \) which can be written in the form (5.41). The replacement can therefore be viewed as a “translation” of integration variables. Let us now choose \( T_{02} \) to be block diagonal in [1, 2] block notation and to be given in block construction by the sequence of 2 \( \times \) 2 matrices \((Q, 1, 1, 1)\) with \( Q \) orthogonal. This choice of \( T_{02} \) induces the transformation \( t_{12} \to \hat{Q}^T t_{12}, t_{21} \to t_{21} \hat{Q} \) with \( \hat{Q} \) a \( 4 \times 4 \) graded matrix with diagonal blocks \((Q, 1)\). Under this transformation, the matrix \( \sigma_2 \) remains unchanged, and \( \sigma_1 \to \hat{Q}^T \sigma_1 \hat{Q} \) which shows that the eigenvalues of \( \sigma_p \) (and, therefore, the “eigenvalues” of \( t_{12} \)) remain unchanged. It follows that the transformation \( T_{02} \) affects only the form of the matrix \( U_1 \) in eqs. (1.18). We have \( U_1 = O_1 v_1 \to U_1 \hat{Q} = (O_1 \hat{Q}) (\hat{Q}^T v_1 \hat{Q}) \). The product \((O_1 \hat{Q})\) can be written as a single orthogonal transformation, the new variable being the sum of the arguments of \( O_1 \) and \( \hat{Q} \). This corresponds to a shift of integration variables which can be absorbed into \( O_1 \). We see that the replacement \( T_{01} = T_{01} - T_{02} \) chosen as specified above amounts ultimately to the substitution

\[
v_1 \to \hat{Q}^T v_1 \hat{Q}.
\]

The result of the calculation must be invariant under the change of integration variables (L.3). The matrices \( v_1 \) occur only in the combination \( v_1 I(1) v_1^{-1} = I(1) v_1^{-2} \). Using the expansion (K.26) and the requirement that the result be invariant under the substitution (L.3) we find that terms of odd order in \( Y_1 \) vanish, and that \( Y_1 \) reduces to the invariant form \(-A_1 I(1)\), with \( A_1 = \hat{\alpha}_1 \hat{\alpha}_1^\dagger + \hat{\beta}_1 \hat{\beta}_1^\dagger \). No conditions are imposed by this symmetry argument on terms involving \( Y_0^0 \) and \( Y_0^1 \). The same reasoning is used to show that in \( v_2 I(2) v_2^{-1} \) only the terms involving \( Y_0^0 \), \( Y_0^4 \), and of terms proportional to \( Y_2^0 \) only the contribution \( -A_2 I(2) \), with \( A_2 = \hat{\alpha}_2 \hat{\alpha}_2^\dagger + \hat{\beta}_2 \hat{\beta}_2^\dagger \), must be kept. It thus follows that we have (see eq. (K.26))

\[
v_p I(p) v_p^{-1} \to I(p) - 2A_p \cdot 1 + 4\hat{\alpha}_p \hat{\alpha}_p^\dagger \hat{\beta}_p \hat{\beta}_p^\dagger \cdot 1, \quad p = 1, 2.
\]

Using the same argument but a different choice of \( T_{02} \) we can show that the terms proportional to \( A_p \cdot 1 \) must vanish also. Instead of an orthogonal matrix, we now choose for \( T_{02} \) the matrix \((W, 1)\). We employ block construction, \( 1 \) is the \( 4 \times 4 \) unit matrix, and \( W \) a graded \( 4 \times 4 \) matrix of the form

\[
W = \exp(Z),
\]

with \( Z \) of the form (K.25), (K.23). Under this transformation, \( t_{21} \to t_{21} W, t_{12} \to W^{-1} t_{12}, \) and \( \sigma_2 \to \sigma_2, \sigma_1 \to W^{-1} \sigma_1 W \). The eigenvalues of \( \sigma_p \) (and, therefore, the “eigenvalues” of \( t_{12} \) and \( t_{21} \)) remain unchanged, but the matrix \( U_1 \) is modified. The modification is calculated most easily by taking \( Z \) to be infinitesimal. Using eq. (1.5) for \( p = 1 \), we find

\[
W^{-1} \sigma_1 W = \begin{pmatrix}
\kappa_1^{-1} + \Xi^+ f_1 - b_1 \Xi^+ & \kappa_1^\dagger + \Xi^+ f_1 - b_1 \Xi^+ \\
\kappa_1 - \Xi b_1 + f_1 \Xi & f_1 - \Xi \kappa_1^\dagger - \kappa_1 \Xi^+
\end{pmatrix}.
\]

Here, \( Z \) is parametrized by \( \Xi \) and \( \Xi^+ \) in complete analogy to eq. (K.25). Substituting for \( D_1 \) and \( N_1 \) in eq. (I.9) the new values given by eq. (L.6), and writing \( \delta \xi_1 \to \xi_1 + \delta \xi_1 \), we find

\[
\delta \xi_1 = \Xi + \cdots
\]

where the dots indicate terms of second order in the Grassmann variables contained in \( \kappa_1 \). It is clear
that $O_1$ is also affected by the substitution $t_21 \rightarrow t_21 W$. However, $O_1$ remains orthogonal after the substitution, and for any orthogonal $2 \times 2$ matrix $O_1$ we have

$$O_1 v_1 I(1) v_1^{-1} O_1^\dagger \rightarrow O_1 (I(1) - 2 A_1 \cdot 1 + 4 \hat{\alpha}_1 \hat{\alpha}_1^\dagger \hat{\beta}_1 \hat{\beta}_1^\dagger \cdot 1) O_1^\dagger = I(1) - 2 A_1 \cdot 1 + 4 \hat{\alpha}_1 \hat{\alpha}_1^\dagger \hat{\beta}_1 \hat{\beta}_1^\dagger \cdot 1 \quad (L.8)$$

where we have used the substitution (L.4). Therefore the modification of $O_1$ is without interest. The substitution $t_21 \rightarrow t_21 W$ ultimately leads to the substitution $\xi_1 \rightarrow \xi_1 + \delta \xi_1$ with $\delta \xi_1$ given by eq. (L.7). We require that terms generated from the r.h.s. of eq. (L.8) by this substitution vanish. Since $A_1 = \hat{\alpha}_1 \hat{\alpha}_1^\dagger + \hat{\beta}_1 \hat{\beta}_1^\dagger$ it is clear that we can generate the second-order term by twofold substitution in the fourth-order term, so that the second-order term must vanish. Unfortunately, the same conclusion cannot be applied directly to the term of zeroth order as this term is proportional to $I(1)$ and thus has another matrix structure than the terms of 2nd and 4th order which are multiples of the unit matrix. Different arguments given at the end of this appendix are used to show that the term of zeroth order vanishes, too. For the time being, we note that the arguments given above apply similarly to the $p = 2$ case, leading to

$$U_p^{-1} O_p v_p I(p) v_p^{-1} O_p^\dagger (U_p^\dagger)^{p-1} \rightarrow I(p) + 4 \hat{\alpha}_p \hat{\alpha}_p^\dagger \hat{\beta}_p \hat{\beta}_p^\dagger \cdot . \quad (L.9)$$

To derive eq. (8.6a) and, along the same lines, eq. (8.7), we deviate from the style used everywhere else in this paper and only sketch the elements of the proof. We do so because we use a theorem due to Wegner [17] which was communicated to us but is as yet unpublished. We first consider the following simple example. Let $F(p, p')$ be a function of a graded vector $p$,

$$p = \left( \begin{array}{c} a \\ \theta \end{array} \right), \quad p^\dagger = (a^*, \theta^*) \quad (L.10)$$

Here, $a$ is complex and commuting, and $\theta$ is anticommuting. We call $F(p, p')$ invariant if it satisfies

$$F(p, p') = F(up, p^\dagger u^\dagger) \quad (L.11)$$

with $u$ a graded unitary (but otherwise arbitrary) matrix. We show that

$$\int F(p, p') da \, da^* \, d\theta \, d\theta^* = F(0, 0) \quad (L.12)$$

if $F$ is non-singular and vanishes at the boundary of the domain of integration. To prove eq. (L.12), we expand

$$F(p, p') = F_0(a, a^*) + F_1(a, a^*) \, \theta + F_2(a, a^*) \, \theta^* + F_3(a, a^*) \, \theta \, \theta \quad (L.13)$$

in powers of the Grassmann variables. Choosing

$$u = \left( \begin{array}{c} 1 \\ \theta \\ \exp(i\varphi) \end{array} \right) \quad (L.14)$$

and using the invariance of $F$, we find $F_1 = 0 = F_2$. Similarly, with
\[
\alpha = \left( \begin{array}{c}
\alpha \\
\alpha^* \\
1
\end{array} \right) + O(\alpha^2)
\]  

we find that

\[
F_3 = \frac{1}{a} \frac{\partial F_0}{\partial a^*} = \frac{1}{a^*} \frac{\partial F_0}{\partial a}.
\]  

Inserting this into eq. (L.13) and the latter into the integral in eq. (L.12), introducing polar coordinates \(a = \rho \exp(i\psi), \ a^* = \rho \exp(-i\psi)\) and carrying out the integration, we obtain eq. (L.12) provided that \(F_0\) vanishes at the upper end of the \(\rho\)-integral. The identity (L.12) for invariant functions of graded vectors can be generalized in several different ways. Parisi and Sourlas [22] have used the straightforward extension of eq. (L.12) to invariant functions of graded vectors with \(d \neq 2\) commuting components. Wegner [17] has derived an analogous integral identity for graded matrices. In addition, he has generalized both identities to functions of several vectors or matrices with a global invariance. (The proof has been carried out for a non-compact domain of integration.) In the present context, we apply a further generalization of the theorem—as yet not proved in its full generality—to eq. (8.6a). The generalization is required by the compactness of the Fermion–Fermion integration manifold. However, the measure \(d\mu(t)\) contains the factor \((1 - \mu^2)\), cf. eq. (K.34). Since \(\mu = 1\) at the upper integration limit, the integrand vanishes at the upper boundary. Moreover, the volume element appearing on the r.h.s. of eq. (K.34) is non-singular for all values of \(\mu_1, \mu_2, \mu_3\), and so is therefore the integrand in eq. (8.6a), if we take the elements of \(t_{12}\) as the independent variables of integration. Finally, the integrand depends only on the eigenvalues of \(t_{12}t_{21}\) and is obviously invariant under graded unitary transformations of \(t_{12}\). We consider \(t_{12}\) as being composed of 4 graded vectors and use the corresponding generalization of eq. (L.12). This yields unity for the value of the integral since \(\mu_1 = 0 = \mu_2\) and \(\mu = 0\) at \(t_{12} = 0\). In the same way, one obtains eq. (8.7). The integral vanishes because of the extra factor \(\alpha_{p}\) multiplying the exponential. In the same way, one finds that terms of zeroth order in \(Y_p\) \((p = 1 \text{ or } 2)\) vanish. This is because the integrand is invariant under the substitution \(t_{21} \rightarrow t_{21} W_1\) \((p = 1)\) or \(t_{12} \rightarrow t_{12} W_2\) \((p = 2)\) where \(W_p\) is graded unitary. Omitting terms of zeroth order, we obtain from the formula (L.9) the substitution rule (8.3).

References

Note added in proof

In section 2 we argue that direct reactions need not be considered as they can always be removed by a unitary transformation. Recently, a new and simple proof of this assertion, tailored to the formulation used in the present paper, has been given [H. Nishioka and H.A. Weidenmüller, Phys. Lett. 157B (1985) 101].

In section 9 we raise a number of questions concerning the connection between our result (8.10), and previous results in compound-nucleus theory. Most of these questions have been answered [J.J.M. Verbaarschot, Ann. Phys. (N.Y.), in press]. In particular, it has been shown that in the limits \( \Gamma \gg d \) and \( \Gamma \ll d \), eq. (8.10) yields the same formulas as earlier work; that in the intermediate domain, a numerical evaluation of eq. (8.10) yields cross-section values which agree with earlier Monte-Carlo calculations; and that there is also complete agreement between such values and the numerical results obtained from the maximum-entropy approach. Equation (8.10) has also been used [J.J.M. Verbaarschot and S. Yoshida, Z. Phys. A, in press] to give an exact expression for the time-development function needed for the analysis of experiments using crystal blocking, and to compare this expression with approximate results obtained earlier.

The arguments sketched in appendix L have been extended and completed [M.R. Zirnbauer, Nucl. Phys. B (FS), in press]. This paper also contains an application of the present formalism to the localisation problem, and a derivation of Efetov's solution for the Cayley tree model.