Chapter 3 of Solitons and Instantons

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where the functional $I_A$ and $A$ stand for the two terms on the right-hand

\[
I_A = \frac{1}{2} \int d^4x \left[ \partial^2 \phi^2 - \frac{1}{2} A \phi^4 \right]
\]

where $A$ is the Lagrangian in $D$ dimensions. The equation is the square of the non-linear field

\[
\phi_i = \frac{\rho_i}{\phi^2}
\]

\[
\frac{d^2}{dt^2} \phi_i = \frac{\rho_i}{\phi^2}
\]

\[
(\phi_i(t)) = \phi(\phi^2 - \lambda^2)
\]

3.2. A Field Theorem

3.3. On solutions in higher dimensions

\section*{Monopoles and Such}
side. Note that not only \( \mathcal{W} \) but also \( V_1 \) and \( V_2 \) are non-negative. Now, let \( \phi(x) \) be a static solution. Consider the one-parameter family of configurations

\[
\phi_t(x) = \psi(x).
\]

(3.4)

It is easy to check that

\[
\mathcal{W}'(\psi) = V_1(\phi) + V_2(\phi) + \frac{\lambda^2}{2} V_1(\phi) + \frac{\lambda^4}{2} V_2(\phi).
\]

(3.5)

Since \( \phi_t(x) \) is an extremum of \( \mathcal{W}(\psi) \), it must in particular make \( \mathcal{W}'(\phi) \) stationary with respect to variations in \( \lambda \); that is,

\[
(\partial/\partial \lambda) \mathcal{W}(\phi) = 0 \quad \text{at} \quad \lambda = 1.
\]

(3.6)

Differentiating (3.5) using (3.6) gives us

\[
(2-D) V_1(\phi) = D V_2(\phi).
\]

(3.7)

Since \( V_1 \) and \( V_2 \) are non-negative, (3.7) cannot be satisfied for \( D \geq 3 \) unless \( V_1(\phi) = V_2(\phi) = 0 \). This means that \( \phi(x) \) has to be space-independent and equal to one of the zeros of \( U(\phi) \). This is just a trivial solution and the theorem precludes non-trivial space-dependent solutions. Note that the result holds only for static solutions and for Lagrangians of the form (3.1). Time-dependent solitary waves can exist for scalar fields in \( (D+1) \) dimensions even when \( D \geq 3 \) (see chapter 8). The theorem does motivate us, however, if we are seeking static solutions in the real \( (3+1) \) dimensional world, to consider models with more than just scalar fields. In section 3.4 we shall study such a model involving scalar and vector fields. Before doing that, it will be helpful to study a simpler system, where some important ideas occurring in later sections are already present. This system involves only scalar fields, but is in \( (2+1) \) dimensions.

3.3. The non-linear \( O(3) \) model: The isotropic ferromagnet

Equation (3.7) tells us that if \( D = 2 \), \( V_2(\phi) = 0 \). That is, \( \phi(x) \), for all \( x \), must be one of the zeros (which are also the absolute minima) of the potential \( U(\phi) \). If \( U(\phi) \) had only discrete minima then, by continuity, \( \phi(x) \) must be the same minimum of \( U(\phi) \) for all \( x \), i.e., it would be independent of \( x \) and trivial. If \( U(\phi) \) had a continuous set of minima, then (3.7) would permit, when \( D = 2 \), a possible \( x \)-dependent solution where \( \phi \) changes continuously within this set of minima.

The simplest example of \( U(\phi) \) with continuous minima is of course \( U(\phi) = 0 \). However this makes the model too simple. The equation obeyed by a static solution, as derived from the Lagrangian (3.1), would be

\[
V^\phi = 0
\]

(3.8)

whose only non-singular solutions are constants. But we can introduce an innocent looking but non-trivial twist to the model by imposing the constraint \( \phi \cdot \phi = 1 \). Such a model is called the non-linear \( O(N) \) model. We will now study the \( N = 3 \) case and find that it does yield interesting solutions (Belavin and Polyakov 1975, Skyrme 1961, Faddeev 1974).

The non-linear \( O(3) \) model consists of three real scalar fields \( \phi(x, t) = [\phi_1(x, t), a = 1, 2, 3] \) with the constraint that at all \( (x, t) \)

\[
\sum_a \phi_a^2(x, t) = \phi \cdot \phi = 1.
\]

(3.9)

The dynamics is determined, subject to the above constraint, by the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \sum_a \left[ \partial_a \phi_a \cdot \left( \partial^a \phi_a \right) - \frac{1}{2} \sum_a \left( \partial_a \phi_a \right)^2 \right].
\]

(3.10)

Note that \( \phi \) can be considered as a vector in an 'internal space' i.e., the three-dimensional field-space, and its components are labelled above by the index \( a \). This is to be distinguished from vectors in coordinate space, which are labelled by Lorentz indices, such as \( \mu \) in (3.10). Thus, in (3.10), a scalar product is implied between \( \partial_a \phi \) and itself, in internal space (as indicated by the dot) as well as in coordinate space as indicated by the repeated index \( a \). We emphasize this because we will be repeatedly dealing with objects that are tensors in coordinate space, as well as in some internal space and the distinction between the two spaces should be carefully borne in mind.

Finally, we note that both the Lagrangian (3.10) and the constraint (3.9) are invariant under global \( O(3) \) rotations in internal space.

The field equation is obtained by applying the Euler-Lagrange variational principle to the action, with the constraint imposed through a Lagrange multiplier. That is, we extremize

\[
S[\phi] = \int \mathcal{L} \, d\tau = \int \left( \sum_a \partial_a \phi_a \cdot \left( \partial^a \phi_a \right) + \lambda (\phi \cdot \phi - 1) \right). \tag{3.11}
\]

The resulting field equation is

\[
\partial_a \left( \partial^a \phi \right) + \lambda \phi = \left( \phi \cdot \phi - 1 \right) \phi = 0. \tag{3.12}
\]
The Lagrange multiplier $\lambda(x, t)$ is eliminated by using the constraint (3.9):
\[\lambda(x, t) = \int \; (\phi(x) - \phi(x)) \; d^3x = 0.\]  
(3.13)

Let us now restrict ourselves to two space dimensions and to static solutions. The field equation then reduces to
\[\nabla^2 \phi - (\phi_0^{-2} \phi_0^3 \phi^{2}) = 0,\]  
(3.14)

upon inserting (3.13).

Equation (3.14) is to be contrasted with (3.8) which would have resulted in the absence of the constraint. We shall see that, unlike (3.8), equation (3.14) does yield interesting non-singular solutions in two dimensions. Furthermore, the solutions can be classified into homotopy sectors, characterised by a topological index. The basic principles behind such classification are just a generalisation of our earlier discussion in the simpler one-dimensional context in chapter 2.

The energy of a static solution, as obtained from the Lagrangian (3.10), is clearly
\[E = \frac{1}{2} \int \left( \nabla \phi \cdot \nabla \phi \right) d^3x, \quad \sigma = 1, 2.\]  
(3.15)

Consider first the zero-energy solutions (the 'classical vacua'). Clearly, they must satisfy for all $x$ the condition $\phi = 0$. That is, $\phi(x) = \phi_0^0$, which is any unit vector in internal space. While $\phi_0^0$ must be $x$-independent in an $E = 0$ solution, it could point in any direction in internal space, as long as it is a unit vector (because of the constraint (3.9)). Thus we have a degenerate continuous family of $E = 0$ solutions, corresponding to the different directions in which $\phi_0^0$ could point. As in the kink system (eq. (2.24)), this is once again a case of 'spontaneous symmetry breaking' at the classical level.

Whereas in the kink problem the symmetry in question was discrete (under $\phi \to -\phi$) and correspondingly there were two discrete $E = 0$ solutions related to one another by that symmetry, here we have continuous O(3) symmetry and correspondingly a continuous family of degenerate classical minima, once again related to one another by O(3) rotations in internal space.

Next, we proceed to soliton solutions, i.e. those with non-zero but finite $E$. From (3.15), it is clear that they must satisfy, using polar coordinates $(r, \theta)$ in $x$-space,
\[r \left| \nabla \phi \right| \to 0 \quad \text{as} \quad r \to \infty,\]  
(3.16)

or
\[\lim_{r \to \infty} \phi(x) = \phi_0^0.\]  
(3.17)

where $\phi_0^0$ is again some unit vector in internal space. Note that as we tend to infinity in coordinate space in different directions, $\phi(x)$ must approach the same limit $\phi_0^0$. Otherwise $\phi(x)$ will depend on the coordinate angle $\theta$ even at $r = \infty$, and the angular component of the gradient, $(1/r)(\partial \phi/\partial \theta)$, will not satisfy (3.16).

Since $\phi(x)$ approaches the same value $\phi_0^0$ at all points at infinity, the physical coordinate plane $R_2$ is essentially compacted into a spherical surface, which we will call $S_2^2$. That is, the plane $R_2$ may be folded into a spherical surface, with the circle at infinity reduced to the north pole of the sphere. (In more precise terms, this can be done by a stereographic mapping; see below.) Meanwhile, the 'internal space', i.e. the space of fields $\phi$, subject to $\sum_{\sigma=1}^{2} \phi^2_\sigma = 1$, is also a spherical surface, of unit radius. Let us call this $S_2^{2m}$. Then, any finite-energy static configuration $\phi(x)$ is just a mapping of $S_2^{2m}$ into $S_2^{2m}$.

Now, we will state (without giving the proof) a result well known in topology. All non-singular mappings of one spherical surface $S_2$ into another $S_2$ can be classified into homotopy sectors. Mappings within one sector can be continuously deformed into one another, whereas mappings from different sectors cannot be continuously deformed into one another. Furthermore, there is a denumerable infinity of such homotopy sectors or classes, which can be characterised by the set of integers, positive, negative and zero. More precisely, these homotopy classes themselves form a group which is isomorphic to the group of integers. Formally all this is written in the compact form
\[n_1(S_2) = Z.\]  
(3.18)

where $n_1(S_2)$ stands for the homotopy group associated with mappings $S_2 \to S_2$, and $Z$ is the group of integers (see the classic text by Steenrod (1951) for a collection of such results and their proof).

We will on several occasions be using results such as (3.18). Readers unfamiliar with them may derive some comfort by considering the simpler case of mapping circles into circles. Consider a circle $S_1$ (characterised by an angle $\theta$ (defined modulo $2\pi$) mapped into another circle $S_1$ (characterised by a $\Lambda$). A mapping is given by a continuous function $\Lambda(\theta)$, modulo $2\pi$. Consider for instance two such mappings or functions
\[\Lambda_\theta(\theta) = 0 \quad \text{for all} \quad \theta\]  
(3.19)

and
\[\Lambda_\theta(\theta) = \begin{cases} \theta & \text{for 0} \leq \theta < \pi \\ (2\pi - \theta) & \text{for \pi} \leq \theta < 2\pi \end{cases}\]  
(3.20)
where \( r \) is some real parameter in the range \([0, 1]\). It is clear that by varying the parameter \( r \) continuously down to zero the second mapping can be continuously deformed into the first. This is also intuitively evident from fig. 6(b) and 6(c). These two mappings therefore belong to the same homotopy class. By contrast, consider the mapping

\[
\Lambda_1(\theta) = \theta \quad \text{for all } \theta.
\]  \( (3.21) \)

This is also a continuous mapping modulo 2\( \pi \), since as \( \theta \) completes a full circle so does \( \Lambda_1 \). However, it cannot be continuously deformed into (3.19) or (3.20). This should be intuitively evident from fig. 6(d), which cannot be distorted into 6(c) or 6(b) without snapping the \( A \)-circle somewhere. The reason obviously is that in (3.21) the second circle is wound once around the first circle, whereas in (3.19) and (3.20) it is effectively wound zero times. Thus (3.21) belongs to a different homotopy class from (3.19) or (3.20). Indeed the integer distinguishing the two classes is just the 'winding number'.

\[
\int_0^{2\pi} \frac{d\Lambda}{d\theta} \quad \theta
\]  \( (3.22) \)

which is clearly zero for (3.19)-(3.20), but equal to unity for (3.21). It is also clear that by winding the second circle an arbitrary number of times one can generate a denumerable infinity of homotopy classes. Thus

\[
\Lambda_\alpha(\theta) = \alpha \theta
\]  \( (3.23) \)

is the prototype mapping belonging to the \( \alpha = n \) class. Negative values of \( \alpha \) are obtained by doing the winding in the opposite sense, for instance by replacing \( \theta \) by \(-\theta\) in (3.23). These qualitative remarks do not constitute a proof, but hopefully they render plausible the result

\[
\pi_1(S_1) = \mathbb{Z}.
\]  \( (3.24) \)

Equation (3.18) is just a generalisation of this result for the mappings \( S_2 \rightarrow S_2 \). The integer characterising the homotopy classes of \( S_2 \rightarrow S_2 \) is again the generalised winding number, i.e. the number of times one of the spheres has been wrapped around the other.

Returning to our O(3) model, in summary, finite-energy static configurations \( \phi(x) \) in two space dimensions can be classified into homotopy sectors, characterised by some integer \( n \) which we will label \( Q \). Note that the nature of the homotopy classification here differs from those in the kink or sine-Gordon problems in chapter 2. In those examples, the different sectors differed in the boundary values of the field at spatial infinity. In the present O(3) model, the only role of the boundary condition (3.17) is to compactify coordinate space into a spherical surface \( S^2 \). True, the boundary condition (3.17) does not uniquely specify the boundary value \( \bar{Q} \). While \( \bar{Q} \) must be the same at all points at spatial infinity, it could point in any 'internal' direction. However, field configurations with different directions of \( \bar{Q} \) can be obtained from one another continuously through O(3) rotations in internal space. Therefore the different choices of \( \bar{Q} \) do not, in themselves, lead to different homotopy sectors for this O(3) model. The sectors arise, instead, from the behaviour of the fields throughout space, including the interior. It will be helpful to write \( Q \) as an integral over the field function \( \phi(x) \), analogous to the integral in (3.22). Such an expression is given by

\[
Q = \frac{1}{8\pi} \int S^2 \phi \cdot (\partial \phi \times \partial_\phi) d^2x
\]  \( (3.25) \)

\( \bar{Q} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Lambda}{d\theta} \quad \theta \) 

\( \int_0^{2\pi} \frac{d\Lambda}{d\theta} \quad \theta \)
where \( \mu = 1, 2 \) and \( v = 1, 2 \). We refer to the coordinate-space indices while the dot and cross product refer to vectors in internal space. That this expression does provide the winding number (also sometimes called the topological index or number) may be seen as follows:

The sphere \( S^{2n} \) in internal space can be described by two variables \( (z_1, z_2) \) such as polar angles, instead of the three cartesian variables \( \phi \), subject to \( \sum \phi_i^2 = 1 \). There is a well-known expression which relates the surface area element as written in terms of cartesian and spherical variables:

\[
dS^{2n} = d^2z \left( \frac{1}{2} \sum_{i=1}^{2n} \frac{\partial \phi_i}{\partial z_i} \frac{\partial \phi_i}{\partial z_i} \right)
\]

(3.26)

Now,

\[
Q = \int \frac{1}{2} \sum_{i=1}^{2n} \frac{\partial \phi_i}{\partial z_i} \frac{\partial \phi_i}{\partial z_i} \d^4z
= \frac{1}{8\pi} \int \sum_{i=1}^{2n} \frac{\partial \phi_i}{\partial z_i} \frac{\partial \phi_i}{\partial z_i} \d^4z
= \frac{1}{8\pi} \int \sum_{i=1}^{2n} \frac{\partial \phi_i}{\partial z_i} \frac{\partial \phi_i}{\partial z_i} \d^4z
\]

(3.27)

since the Jacobian of the change of variables from \( (x_1, x_2) \) to \( (z_1, z_2) \) is given by

\[
\varepsilon_{x_i} d^4x = \varepsilon_{z_i} d^4z
\]

(3.28)

Inserting (3.26) into (3.27), we have

\[
Q = \frac{1}{4\pi} \int dS^{2n} \phi = \frac{1}{4\pi} |dS^{2n}|
\]

(3.29)

since \( \phi_i \) is just a unit vector normal to the surface. Recalling that \( S^{2n} \) is the surface of a unit sphere, with area \( 4\pi \), we clearly see that \( Q \) gives the number of times the internal sphere is traversed as we span the coordinate space \( S^2 \) as compacted into \( S^{2n} \).

This homotopy classification is valid for any static field configuration for which the energy functional (3.15) is finite. It does not require that the fields be solutions of the field equation (3.14). Of course, finite-energy solutions are subsets of finite-energy configurations, and the same classification holds for them. In order to actually find some solutions in any given \( Q \)-sector, we employ an ingenuous trick (Belavin and Polyakov 1973).

The non-linear O(3) model: The isotropic ferromagnet

We begin with the identity

\[
\int d^4z \left[ (e_{\mu} \phi \pm e_{\mu} \phi \times \partial_\mu \phi) + (e_{\mu} \phi \pm e_{\mu} \phi \times \partial_\mu \phi) \right] \geq 0.
\]

(3.30)

The identity holds since the integrand is just the scalar product of the vector in brackets with itself. Upon expanding, this becomes

\[
\int d^4z \left[ (e_{\mu} \phi \cdot (\partial_\mu \phi)) + (e_{\mu} \phi \cdot (\partial_\mu \phi)) + (e_{\mu} \phi \cdot (\partial_\mu \phi)) \right] \geq 2 \int d^4z \left[ (e_{\mu} \phi \cdot (\partial_\mu \phi) \right].
\]

The two terms on the left-hand side are actually equal to each other since

\[
e_{\mu} \phi \cdot (\partial_\mu \phi) (\partial_\mu \phi) = (\partial_\mu \phi) (\partial_\mu \phi) + (\partial_\mu \phi) (\partial_\mu \phi)
= (\partial_\mu \phi).
\]

where we have used the constraint \( \phi \cdot \phi = 1 \) and its derivative \( \phi \cdot e_{\mu} \phi = 0 \). Thus,

\[
2 \int d^4z (\partial_\mu \phi) (\partial_\mu \phi) \geq 2 \int d^4z (\partial_\mu \phi) (\partial_\mu \phi)
\]

or

\[
E \geq 4\pi |Q|.
\]

This inequality sets a lower bound for the energy of any static configuration in a given \( Q \)-sector. Now, the static field equation (3.14) is obtained by extremizing the static energy functional (3.15), subject to the constraint (3.9). Since a configuration from one sector cannot, under continuous variation, move into another sector, the extremization can be done in each sector separately. In any given \( Q \)-sector, the energy is minimised when the equality (3.31) is satisfied. This in turn implies that the equality (3.30) is satisfied, which will happen if and only if

\[
e_{\mu} \phi = \pm \varepsilon_{\mu} \phi \times (\partial_\mu \phi).
\]

(3.32)

Any field configuration that satisfies (3.32) as well as the constraint (3.9), will minimise \( E \) in some \( Q \)-sector, and will therefore automatically satisfy the extremum condition given by the field equation (3.14). This can be explicitly verified. For any configuration satisfying (3.32),

\[
\nabla \phi = \varepsilon_{\mu} \phi \times (\partial_\mu \phi)
\]

\[
= \pm \varepsilon_{\mu} (\partial_\mu \phi) \times (\partial_\mu \phi)
= \varepsilon_{\mu} (\partial_\mu \phi \times (\partial_\mu \phi)
\]
\[ \phi \cdot \theta \phi = 0 = \Phi \cdot \theta \Phi + \Phi \cdot \theta \Phi \]

which is just the field equation. In the last step, we have used

\[ \phi \cdot \theta \phi = 0 = \Phi \cdot \theta \Phi + \Phi \cdot \theta \Phi \]

which result from differentiating the constraint (3.9). Even though any field that satisfies (3.32) will also satisfy (3.14), the converse need not be true. One could in principle have solutions of (3.14) which do not satisfy (3.32). These would not represent absolute minima of \( E \) in the corresponding \( Q \)-sector, but some higher valued extrema of \( E \), as such local minima. We will remain content with finding solutions of (3.32). Note that this is an easier equation to solve than the parent field equation (3.14). It is a first-order differential equation while the latter is a second-order equation.

In fact (3.32) can be further simplified by a change of variables. We recall that the allowed values of \( \phi \), subject to \( \phi \cdot \phi = 1 \), form the surface of the unit sphere, \( S^{2n-1} \). Let us stereographically project the surface of this sphere onto a plane, and represent points on the former by giving the cartesian coordinates \( \omega_1 \) and \( \omega_2 \) of the corresponding points on the plane. The variables \( \omega_1 \) and \( \omega_2 \) are related to the variables \( \phi \) by

\[ \omega_1 = 2\phi_1/(1 - \phi_3) \quad \text{and} \quad \omega_2 = 2\phi_2/(1 - \phi_3) \quad (3.33) \]

where the plane on which the projection is made is parallel to the \( \{ \phi_1, \phi_2 \} \) plane and contains the south pole. It will also be useful to construct the complex quantity

\[ \omega = \omega_1 + i\omega_2 \quad \text{where} \quad \phi = \phi_1 + i\phi_2. \]

Then

\[ \partial_1 \omega = \frac{\partial \omega}{\partial \phi_1} - 2(1 - \phi_3^2)\phi_1 + \phi \phi_1)/(1 - \phi_3)^2 \]

\[ = [2/(1 - \phi_3^2)] \partial_1 \phi_1 + \phi \phi_2 \phi_3. \quad (3.34) \]

Now, equation (3.32) tells us that

\[ \partial_1 \phi = \pm i\partial_1 \phi_2 \quad \text{and} \quad \partial_2 \phi = \pm i\partial_2 \phi_1. \]

Substituting this into (3.34), we have

\[ \partial_1 \omega = \pm i\partial_1 \omega. \quad (3.35) \]

In terms of \( \omega_1 \) and \( \omega_2 \), this gives

\[ \frac{\partial \omega_1}{\delta x_1} = \frac{\partial \omega_2}{\delta x_2} \quad \text{and} \quad \frac{\partial \omega_2}{\delta x_2} = -\frac{\partial \omega_1}{\delta x_1}. \quad (3.37) \]

We remind the reader: \( x_1 \) and \( x_2 \) are cartesian coordinates of our original two-dimensional physical space; \( \omega_1 \) and \( \omega_2 \) describe the plane in 'internal space' on which \( S^{2n-1} \) has been projected stereographically.

Equation (3.37) is all too familiar as the Cauchy–Reimann condition for \( \omega \) being an analytic function of \( z \) (for the upper signs) or \( \bar{z} \) (for the lower signs), where \( z = x_1 + ix_2 \). Thus, any analytic function \( \omega(z) \) or \( \omega(z^*) \) automatically solves (3.32) and therefore also the field equation, when written in terms of the original variables \( \phi_1 \) and \( \phi_2 \). Furthermore, while \( \omega \) must be analytic in either \( z \) or \( z^* \), it need not be an entire function. While cuts are prohibited by the single-valuedness of \( \Phi(z) \), isolated poles in \( \omega(z) \) are permitted. That \( \omega \) diverges at the poles need not cause concern, \( \omega \to \infty \) merely corresponds to the 'north pole' in \( S^{2n-1} \), i.e. to \( \phi_3 = 1 \).

It will be useful to write down the expressions for \( E \) and \( Q \) in terms of \( \omega \) for the case when \( \omega \) is analytic in, say, \( z \). They are given by (the derivation is straightforward)

\[ E = \int d^2 x \frac{|d\omega| |dz|}{(1 + |\omega|^2/4)^3} \quad \text{and} \quad |Q| = \frac{E}{4\pi}. \quad (3.38) \]

A prototype solution for arbitrary positive \( Q \) is given by

\[ \omega(z) = \left( x - \frac{z_0}{|z|^2} \right)^n \quad (3.39) \]

where \( n \) is any positive integer, \( x \) is any real number and \( z_0 \) is any complex number. Since (3.39) is an analytic function, our analysis assures us that it will yield an exact static solution of the field equation when rewritten in terms of \( \phi_1 \) and \( \phi_2 \). Upon Lorentz transformation into a moving frame, it will yield exact time-dependent solutions which move undistorted in shape.

Note that the parent theory is Lorentz-invariant.

In (3.39), \( \omega \) represents a point in field space, while \( z \) stands for a point in coordinate space. Clearly (3.39) allows a roots for \( z \), for a given \( \omega \). Therefore it must correspond to the \( Q = n \) sector. This may be verified by substituting (3.39) into (3.38). We have

\[ Q = \frac{1}{4\pi} \int d^2 x \frac{\pi^2 |x - z_0|^2}{(1 + |\omega|^2/4)^3}. \quad (3.40) \]
Using
\[ z - z_0 = pe^{\phi} \quad \text{and} \quad d^2 \phi = \rho d\rho d\theta \]  
(3.41)
the integration is trivial and yields \( Q = \pi \). Hence \( E = 4\pi Q = 4\pi \alpha \) is finite. 

Clearly, these are explicit solitary-wave solutions, for any positive integer \( n \).

The constants \( \lambda \) and \( \xi_0 \) (which stands for a pair of coordinates \((x_1, x_2)\)) refer to the size and location of the soliton solution. The fact that the solution exists for arbitrary \( \lambda \) and \( \xi_0 \) and the fact that neither \( Q \) nor \( E \) depend on these constants is a reflection of scale and translational invariance. Notice that \( \text{E} \{ \phi \} \) in (3.15) is obviously invariant under \( x \rightarrow x - a \) and \( x \rightarrow x + b \).

The O(3) model is also interesting in \((1 + 1)\) dimensions. It has been shown (Pohlmeier 1976, Luscher and Pohlmeier 1978) that, like the sine–Gordon system, the O(3) model in \((1 + 1)\) dimensions is also characterised by an infinite number of conserved quantities and by Backlund transformations for generating solutions. [In the quantised version of the theory, it has been shown that it is asymptotically free and that the conserved quantities exist free of anomalies (Luscher 1978, Polyakov 1977a).] Finally, an exact factorised S-matrix has been constructed using the existence of these infinite conserved quantities (Zamolodchikov and Zamolodchikov 1979). We shall discuss this last aspect in a later chapter.

The static solutions we obtained in (2 + 1) dimensions are also relevant to the model in \((1 + 1)\) dimensions. They serve as instantons of the latter (see the next chapter). Lastly, this O(3) model and its solutions are also relevant in describing the statistical mechanics of an isotropic ferromagnet (see section 3.6).

3.4. The 't Hooft–Polyakov monopole

We are finally ready to discuss static soliton solutions for the realistic case of \((3 + 1)\) dimensions. The virial theorem of section 3.2 tells us that spin-zero fields alone cannot yield such a solution; higher-spin fields must be involved. We will not consider spin-\( j \) fields at this stage. They are Fermi fields, whose 'classical limit' requires special interpretation and will be discussed in chapter 9. Therefore we go on to spin-1 fields, in particular to gauge fields, which have attained considerable importance in particle physics.

The simplest example of a gauge theory in \((3 + 1)\) dimensions is the free electromagnetic system whose gauge group is \( U(1) \). This involves only linear (Maxwell's) equations, and is trivial to solve. As it is well known, unlike the \((1 + 1)\) dimensional linear wave equation (2.1), free Maxwell's equations in \((3 + 1)\) dimensions do not yield solitary waves. Any localised packet will necessarily dissipate itself. When the electromagnetic field is coupled to charged scalar fields, soliton solutions do emerge, but in \((2 + 1)\) dimensions.

We will return to this case later. If we shift our interest to non-abelian gauge groups, then the simplest candidate is \( SU(2) \). This group calls for a triplet of gauge fields known as the Yang–Mills fields (Yang and Mills 1954, Shaw 1955). This triplet is self-coupled to form a non-linear system which is far from trivial, but it has been shown that a set of pure Yang–Mills fields in \((3 + 1)\) dimensions also fails to yield any solitary waves (Coleman 1977b, Deser 1976, Pagels 1977). It does yield interesting singular solutions (Yang and Wu 1968) but we are concerned only with non-singular finite-energy solutions.

Therefore we enlarge the system further by coupling the Yang–Mills fields to a triplet of scalar fields as was done by Georgi and Glashow (1972). It has been shown through the pioneering work of 't Hooft (1974b) and Polyakov (1974) that this model does yield a non-singular localised static solution with some remarkable properties. We devote this section to a discussion of this model and its static soliton solution. Non-abelian gauge theories, of which the model under question is an example, have been under study for over twenty years. Although our presentation will be reasonably self-contained it would be helpful if the reader were familiar with the basics of such theories. (There are several good review articles and books, e.g. Abers and Lee (1973), Bernstein (1974), Taylor (1976) and Faddeev and Slavnov (1980).)

The model consists of scalar fields \( \phi^a(x, t) \) and vector fields \( A^a_\mu(x, t) \) in \((3 + 1)\) dimensions. The indices \( a = 1, 2, 3 \) is a collection of fields, which will transform according to local (space-time dependent) \( SU(2) \) transformations given below. For any given \( a, \phi^a \) is a scalar and \( A^a_\mu(x) \) is a vector under Lorentz transformations. The Lagrangian density is

\[ \mathcal{L}(x, t) = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{4} (D_\mu \phi^a)(D^\mu \phi^a) - \frac{1}{2} (\phi^a \phi^a - F^a)^2 . \]  
(3.42)

Here, the 'field tensor' \( G^a_{\mu\nu} \) is defined by

\[ G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \]  
(3.43)

and the 'covariant derivative' \( D_\mu \phi^a \) by

\[ D_\mu \phi^a = \partial_\mu \phi^a + g \epsilon^{abc} A^b_\mu \phi^c . \]  
(3.44)
(The covariant derivatives of other triplets of fields such as $A_{\mu}^3$ or $G_{\mu}^a$ are defined analogously.) The real constants $\mu$, $\lambda > 0$, and $F$ are parameters of the model. The $A_{\mu}^{a}$ are called gauge fields and the $\phi^a$ in such a context, are called Higgs fields since they lead to the so-called Higgs mechanism (see section 10.2).

To readers unfamiliar with non-abelian gauge theories, this Lagrangian may look complicated and contrived. It has however been designed to be 'gauge invariant', i.e., invariant under the set of independent SU(2) transformations at each space-time point. These transformations are defined by

$$\phi^a(x,t) \rightarrow [U(x,t)]_x \phi^a(x,t)$$

(3.45)

and

$$(L^a A_\mu^a)_{\mu} \rightarrow U_{\mu} [L^a A_\mu^a + i(hg)_i \partial_i]_{\mu}(U^{-1})_{\nu}$$

(3.46)

where

$$[U(x,t)]_x = \exp \{-iL^a \phi^a(x,t)\}$$

(3.47)

is any member of the group SU(2) at each space-time point $(x,t)$, written here in its $3 \times 3$ representation. $(L^a)_{\mu} = i(hg)_i \nu$ are the three generators of SU(2) in $3 \times 3$ matrix representation. $I$ is the identity matrix and the group parameters $\phi^a$ vary in space-time. One can verify by direct substitution that the Lagrangian (3.42) is invariant under these gauge transformations. These equations are generalisations to the group SU(2) of corresponding statements in electrodynamics for which the gauge group is U(1).

In this section we merely discuss some static classical solutions of this system--an impressive problem in its own right considering that 15 coupled non-linear fields are involved in (3 + 1) dimensions.

The equations of motion that flow from the Lagrangian (3.42) are

$$D_\mu G^{\mu \nu} = -i\mu \left( D^\nu \phi^a \right) g^{\nu \mu} + 2F^2 \phi^a$$

(3.48)

and

$$D_\mu D^\mu \phi^a = -i\phi^b \left( D_\mu \phi^b \right) g^{\mu \nu} + 2F^2 \phi^a$$

(3.49)

We shall restrict ourselves to solutions that are (i) static and (ii) satisfy $A_\mu^3(x) = 0$ for all $x, t$. For these, the field equations reduce to

$$D_\mu G^{\mu \nu} = -i\mu \left( D^\nu \phi^a \right) g^{\nu \mu} + 2F^2 \phi^a$$

(3.50)

and

$$D_\mu D^\mu \phi^a = -i\phi^b \left( D_\mu \phi^b \right) g^{\mu \nu} + 2F^2 \phi^a$$

(3.51)

where $\mu = 1, 2, 3$ are purely spatial indices.

We are looking for finite-energy solutions of (3.50)-(3.51), which are more complicated than the field equations we have encountered so far, but our approach to the problem will be the same as before; that is, we will first find the classical vacuum (zero-energy) solutions. This information will then be used to identify the set of allowed boundary conditions that any finite-energy configuration must satisfy. Next, we will make a homotopy classification of these boundary conditions. Lastly, amongst configurations of a given homotopy sector, we will look for a finite-energy solution. First we look for zeroes of the energy. The expression for the conserved energy of the system can be obtained from the Lagrangian as usual and reduces, for static solutions with $A_\mu^a = 0$, to

$$E = \int d^4x \left\{ 4G_{\mu \nu} G^{\mu \nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a + \frac{1}{4} (\phi^a \phi^b - F^2)^2 \right\}$$

(3.52)

This energy reaches a minimum and vanishes when

$$A_\mu^a(x) = 0$$

(3.53)

$$\phi^a(x) \phi^a(x) = F^2$$

(3.54)

and

$$D_\mu \phi^a = 0$$

(3.55)

which amounts to $\partial_{\mu} \phi^a = 0$ in view of (3.53). (In fact the first condition (3.53) is sufficient, but not necessary. There are several other solutions for $A_\mu^a$ related to $A_\mu^a = 0$ by gauge transformations, which also lead to $E = 0$. We will return to this question in the next chapter. For the moment let us take $A_\mu^a = 0$, and concentrate on the conditions (3.54)-(3.55) on $\phi^a$.)

The conditions on the $\phi^a$ are similar to those in the D(3) model in the last section. There is a degenerate continuous family of $E = 0$ solutions. In each of these, $\phi \equiv (\phi^a)$ must have fixed magnitude $F$, but can point in any $x$-independent direction in internal space. Recall that the local SU(2) gauge symmetry of this system contains in it global ($x$-independent) SU(2) symmetry which in turn amounts to internal rotational symmetry for our real scalar fields $\phi$. The family of $E = 0$ solutions permitted by (3.53)-(3.55) are related to each other by this symmetry.

Let us move on to configurations with non-zero but finite energy $E$. The similarity with the non-linear O(3) model still persists to some extent, but now there are important differences. The condition for finite $E$ is, as before, that the fields approach some $E = 0$ configuration at spatial infinity sufficiently fast. We can see from (3.52) that this condition for the field $\phi$ is,
as \( r = |x| \to \infty \),

\[ r^{-2} \nabla^2 \phi \to 0 \]  

and

\[ \phi \to \pm \infty. \]  

(3.56)

(3.57)

As in the O(3) model, the magnitude of \( \phi \) must approach the 'vacuum' value \( F \). But unlike the O(3) model, the internal space direction of \( \phi \) need not be the same when we go to spatial infinity in different directions. This is because, unlike eq. (3.16) in the O(3) model, the corresponding condition (3.56) here requires the vanishing of the covariant derivative \( \nabla \phi \) and not the ordinary derivative \( \partial \phi \). Consider the expression (3.44) for the covariant derivative, and express it in terms of spherical polar coordinates \((r, \theta, \phi)\) and corresponding components. The \( \theta \)-component of \((\nabla \phi)\) is given by

\[ (\nabla \phi)_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \text{grad}_{\theta} A \nabla \phi. \]  

(3.58)

As long as this combination falls off fast enough to satisfy (3.56), \( \partial \phi/\partial \theta \) itself need not vanish as \( r \to \infty \). A non-zero \( \partial \phi/\partial \theta \) is permitted as \( r \to \infty \), as long as \( A \) is the \( \theta \)-component of the gauge field, matched with \( \phi \nabla \phi/\partial \theta \) in such a way that the combination (3.58) goes to zero as \( r \to \infty \). This in turn

implies that \( A \) falls off to zero only as fast as \( 1/r \). A similar statement obviously holds for the azimuthal components of \((\text{grad} \phi)\) and \( A \). The fact that some components of \( A \) fall off only as slowly as \( 1/r \) as \( r \to \infty \) is consistent with the finiteness of \( E \). Note that the integrand in \( E \) involves \( G_{\jmath} G^\jmath \), which will then decrease as \( 1/r^2 \) and will be integrable.

Therefore unlike the O(3) model, in this case finiteness of \( E \) permits fields where \( \phi(x) \) points in different internal directions at different points on the boundary of space, as long as \( \sum \phi \nabla \phi = 0 \). Thus, the allowed values of \( \phi \) at the boundary lie on a spherical surface of radius \( F \) in internal space. Let us call this surface \( S^\infty_0 \). Space is three-dimensional in this model, so that its boundary is another spherical surface \( S^\infty_0 \). Hence, the set of boundary conditions on \( \phi \) permitted by finiteness of \( E \) are the set of all non-singular mappings of \( S^\infty_0 \) to \( S^\infty_0 \). We have already observed that such mappings fall into a denumerable infinity of homotopy classes which form the group \( \pi_3 (S^3) = Z \). Field configurations from one sector cannot be continuously deformed into another sector. Each sector can be characterized by an integer \( Q \), which is the topological number of this model, and describes the number of times \( S^\infty_0 \) is covered when \( S^\infty_0 \) is traversed once. In a \( Q \neq 0 \) configuration, the field \( \phi \) will either tend to the same value, as \( x \to \infty \) in any direction (fig. 7(a)), or will tend to some angle-dependent value that can be deformed to as to be angle-independent. The trivial vacuum solution \( \phi(x) = \delta \phi / \partial \theta \) belongs to the \( Q = 0 \) sector. These are analogues of functions like (3.19)-(3.20) in the \( S_1 \to S_1 \) case. A prototype \( Q = 1 \) configuration would have \( \phi(x \to \infty) \) pointing radially outward, i.e., with its internal directions parallel to the coordinate vector (fig. 7(b)). This would be the analogue of the mapping (3.21). We will give an explicit example of an \( Q = 1 \) solution later on.

Note that the homotopy classification of finite-energy configurations of this model arise entirely from the boundary conditions on the fields. In this respect, this classification differs from that in the O(3) model (although both involve the same homotopy group \( \pi_3 (S^3) \)), and is closer in spirit to the \( \delta \) and sine-Gordon models. In the latter examples, space was one-dimensional, and the different sectors corresponded to the different values that the field could take at \( x \) approached \( + \infty \) or \( - \infty \). In the kink solution for instance, \( \phi(x) \) approached different limits as \( x \to \pm \infty \). A similar thing happens in the \( Q \neq 0 \) sectors of our gauge model. As we approach \( r \to \infty \), but in different directions, the field \( \phi \) approaches different limits.

![Fig. 7. Cross sections of prototype \( n = 0 \) and \( n = 1 \) mappings of \( S^\infty_0 \to S^\infty_0 \) are given in (a) and (b). The dotted surface stands for \( S^\infty_0 \) and the full arrows give the directions of the corresponding points on \( S^\infty_0 \), i.e., the directions of the vector \( \phi \). Notice that if \( \phi \) were rotated in (b), along the dashed arrows, in an attempt to point them all upwards as in (a), the rotation angle would be discontinuous at the south pole. This is a pictorial way of understanding why the two mappings belong to different homotopy sectors.](image)
We have been restricting ourselves to static configurations with $A_i(x) = 0$. The solution we obtain later on will also come into this category. However, given a static solution with $A_i(x) = 0$, one can obtain others that are time-dependent and carry nonzero $A_i$, by employing the gauge transformations (3.45)-(3.46). Since the Lagrangian is invariant and the field equations covariant under these gauge transformations, the transformation of a solution will also be a solution. It will carry the same energy and topological charge, since the energy is gauge invariant and so, as we will see, is $Q$.

As we did in the earlier models, we can again write the topological index as an explicit functional of the field. In fact, one can define a topological current $k_\phi$ (Arratia et al. 1975) by

$$k_\phi = (1/8\pi_\epsilon_{\mu
u\rho\sigma} \partial^\mu \partial^\nu \phi \partial^\rho \partial^\sigma \phi^* \phi),$$

where

$$\phi = \phi(\phi); \quad |\phi| = \left(\sum \phi^* \phi\right)^{1/2}.$$

From the antisymmetry of $\epsilon_{\mu
u\rho\sigma}$ it follows that

$$\partial^\mu k_\phi = 0.$$

It is clear that the conservation of this current follows just from the construction of $k_\phi$ in (3.59) and not from the dynamics, i.e. the explicit form of the Lagrangian or equations of motion. The same was true of the current (2.59). This is again a reflection of the fact stressed in the last chapter that topological charges and currents are different from the familiar Noether currents and charges. The conserved charge corresponding to (3.59) is

$$Q = \int d^4x k_\phi$$

$$= \frac{1}{8\pi} \int \epsilon_{i\alpha} \epsilon^{\alpha\beta\gamma\delta} \partial_\beta \phi \partial_\gamma \phi^* \partial_\delta \phi \partial_\lambda \phi^* d^4x$$

$$= \frac{1}{8\pi} \int \epsilon_{i\alpha} \epsilon^{\alpha\beta\gamma\delta} \partial_\beta \phi \partial_\gamma \phi^* \partial_\delta \phi \partial_\lambda \phi^* d^4x$$

$$= \frac{1}{8\pi} \int \frac{d^4x}{s F_{\mu\nu}} F_{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* d^4x,$$  \hspace{1cm} (3.61)

where $i, j, k = 1, 2, 3$ are space-indices and $s F_{\mu\nu}$ is our sphere at infinity in coordinate space. To see that (3.61) is in fact the topological mapping index for $S^{(2n)} \rightarrow S^{(n)}$, let us introduce parameters $(x_i, x_j)$ to describe the surface $S^{(n)}$. They could for instance be polar and azimuthal angles. We use the identity (3.26) rewritten for our coordinate space:

$$d^3x_i = \frac{1}{2} d^3x \epsilon^{\mu\nu\rho} \frac{\partial x^\mu}{\partial x_i} \frac{\partial x^\nu}{\partial x_j} \frac{\partial x^\rho}{\partial x_k}; \quad p, q = 1, 2.$$

Also,

$$\partial_i \phi^* = \frac{\partial \phi^*}{\partial x_i} \frac{\partial x_i}{\partial x_k}.$$

Hence,

$$Q = \frac{1}{8\pi} \int d^4x \left(\frac{1}{2} \epsilon_{i\alpha} \epsilon^{\alpha\beta\gamma\delta} \frac{\partial x^\mu}{\partial x_i} \frac{\partial x^\nu}{\partial x_j} \frac{\partial x^\rho}{\partial x_k} \frac{\partial x^\sigma}{\partial x_l} \right)$$

$$= \frac{1}{8\pi} \int \frac{d^4x}{s F_{\mu\nu}} \epsilon_{i\alpha} \epsilon^{\alpha\beta\gamma\delta} \frac{\partial x^\mu}{\partial x_i} \frac{\partial x^\nu}{\partial x_j} \frac{\partial x^\rho}{\partial x_k} \frac{\partial x^\sigma}{\partial x_l}.$$

Equation (3.63) has exactly the same form as (2.23). We need only to repeat the subsequent steps in section (3.3) to see that $Q$ gives the desired topological winding number.

Our next step is to explain why the name 'monopoles' is used for solutions of this model. It will be seen that one can associate a magnetic monopole charge with these solutions. Further, the monopole charge is just proportional to $Q$, the topological number characterizing the solution.

The subject of magnetic monopoles has a long history, both in terms of theoretical studies (Dirac 1931, 1948, Schwinger 1966) and experimental searches. We will not delve into this literature here other than to note the following well-known statement. In conventional electrodynamics, Maxwell's equations for the vector potential $A_\mu$ read

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu$$

where

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} = 0.$$  \hspace{1cm} (3.66)

and where $j^\nu$ is the electric current. Introduction of magnetic charges and currents requires that a magnetic current term be added by hand to the
right-hand side of eq. (3.66). In our non-Abelian SU(2) gauge theory, such a magnetic current term can be shown to be already present without having to alter the Lagrangian (3.42) or the field equations (3.48)-(3.49).

To start with, we had made no physical connection between electromagnetism and the gauge model (3.42), other than to note that as a theory (3.42) is a generalization to the SU(2) group of what happens in electromagnetism, which is a U(1) gauge theory. However U(1) is a subgroup of SU(2), and it is possible to imbed an electromagnetic system as part of the richer system (3.42). At first glance, one might try to associate one of the three gauge fields, say $A^e_\mu$, with the electromagnetic field. But this would not be a gauge-invariant association since the different $A^e_\mu$ mix into one another under the gauge transformations (3.46). 'tHooft therefore presented a gauge-invariant definition for the electromagnetic field tensor $F_{\mu\nu}$ in terms of the parent fields:

$$F_{\mu\nu} = \phi^a G^a_{\mu\nu} - \frac{1}{(g/\mu)^2} \partial_\mu \phi D_\nu \phi D_\nu \phi^a;$$  
(3.67)

It can be verified that (i) this expression is gauge invariant, and (ii) it does reduce in regions where $\phi^a = (0, 0, 1)$ to

$$F_{\mu\nu} = \partial_\mu A^e_\nu - \partial_\nu A^e_\mu.$$  
(3.68)

The second statement says that if in a particular gauge $\phi^a$ always points in the same internal direction, then the vector field along that direction may be considered as the electromagnetic field. In general, (3.67) gives the field tensor. Unlike the usual electromagnetic system (3.64)-(3.66), the tensor in (3.67) has a dual with non-zero divergence. A little bit of algebra shows that (3.67) gives

$$\frac{1}{4} F_{\mu\nu} \Box F^{\mu\nu} = (1/2g) \phi \nabla^2 \phi \nabla^2 \phi \nabla^2 \phi.$$  
(3.69)

where $k_\mu$ is nothing but the topological current (3.59). By comparing with the electric current in (3.64), we see here that the magnetic current is $1/gk_\mu$.

The magnetic field defined as usual by $B = \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$ satisfies

$$\nabla \cdot B = 4\pi k_\mu/(g).$$  
(3.70)

Hence, the total magnetic monopole charge is equal to

$$m = \int_{\mathbb{R}^3} \frac{k_\mu}{g} \, d^3x = Q/g;$$  
(3.71)

where $Q$ is the topological charge.

The 'tHooft-Polyakov monopole

Notice that (3.71) is similar to the Schwinger quantization condition on the allowed values of the magnetic charge (Schwinger 1966). It is, however, not precisely the same as the Schwinger condition, nor should it be. The latter is derived from quantum theory whereas our considerations have so far been strictly classical. The Schwinger condition is

$$m = nh/q; \quad n = 0, 1, 2, \ldots,$$  
(3.72)

where $h = h/2\pi$ is Planck's constant and $q$ is the electric charge of the electron. By contrast, in (3.71) $h$ never appears and $g$ is just a coupling constant in the Lagrangian (3.42)-(3.44). It is only if some quantum theoretic condition forced the electric charge of the particles related to the field $\phi^a$ to be $q = gh$ that our condition (3.71) would become the Schwinger condition.

Since $F^{\mu\nu}$ in (3.67) is gauge-invariant by construction, so are $k_\mu$ and the magnetic charge. The reader may be concerned that while (3.69) gives a non-zero divergence to the dual of $F^{\mu\nu}$, the form (3.68) into which $F^{\mu\nu}$ can sometimes be cast is a pure curl and has vanishing divergence for its dual. However, (3.68) follows from (3.67) only in regions where $\phi^a$ points in the same internal direction. Our discussion on homotopy classes tells us that starting from a $Q \neq 0$ solution, $\phi^a$ cannot everywhere be rotated through non-singular gauge transformations to point in the same direction, for it would then end up in a $Q = 0$ configuration. Therefore $Q \neq 0$ configurations cannot be written everywhere in the form (3.68) and they must have a non-zero dual-divergence in some regions, consistent with the existence of a non-zero magnetic charge.

These general considerations are explicitly illustrated in the $Q = 1$ example developed by 'tHooft and Polyakov. Consider the following ansatz for a static solution:

$$\phi(x) = \delta_{\mu}(x/r) F(r)$$  
(3.73)

$$A^t(x) = \epsilon_{tui}(x/r) W(r); \quad A^u(x) = 0 \quad (3.74)$$

where $r = |x|$ and $i, j, u = 1, 2, 3$. Let $F(r)$ and $W(r)$ be subject to the boundary conditions:

$$F(r) \to F \quad \text{and} \quad W(r) \to 1/qr$$  
(3.75)

as $r \to \infty$. The functions $F(r)$ and $W(r)$ have to be such that the field equations (3.50)-(3.51) are satisfied. $F$ is the same constant that appears in the Lagrangian. We see that with (3.75) the ansatz satisfies all earlier requirements including the boundary conditions (3.56)-(3.57). It is evident
that (3.73)–(3.74) gives a \( Q = 1 \) configuration. In (3.73), the internal direction of \( \phi^r \) is parallel to the coordinate direction \( x^r \) as in fig. 7(b). This is clearly a one-to-one mapping of \( S^2 \) into \( S^3 \). It should correspond to \( Q = 1 \). This can be checked explicitly by inserting (3.73) into the defining integrals (3.61) for \( Q \). Furthermore one can calculate the magnetic field \( B_1 = \frac{1}{r} \frac{e_a}{F} \) as \( r \to \infty \), corresponding to this configuration. Since \( D_1 \phi^r \to 0 \) as \( r \to \infty \), the definition of \( F_1 \) in (3.67) yields quite easily, when (3.73)–(3.75) are inserted, the result

\[
B(r) = \frac{k}{r} \sinh(kFr).
\]

(3.76)

This clearly corresponds to a magnetic pole of strength \( 1/r \) in conformity with \( Q = 1 \). Since \( A_{1r} = 0 \) and all fields are time-independent, \( F_{\mu \nu} = 0 \) and the electric field vanishes. This solution therefore carries only magnetic and no electric charge.

Notice that we have yet to solve explicitly the field equations (3.30)–(3.51). This is because these equations, like most coupled non-linear equations in three space dimensions, are not easy to solve. The bulk of our discussion has been a topological analysis based solely on boundary conditions. It is remarkable that so much information could be extracted about some features of the solutions without actually solving the equations. In particular, for the \( Q = 1 \) ansatz (3.73)–(3.74), we have so far exploited only its special tensorial form and the boundary conditions (3.75). To obtain the functions \( F(r) \) and \( W(r) \), one must substitute the ansatz into the field equations. The ansatz is 'spherically symmetric', i.e. except for simple tensorial factors explicitly shown it is dependent only on the radial variable \( r \). Thanks to this simplification, the partial differential equations (3.30)–(3.51) reduce to ordinary differential equations, for this particular ansatz. On substituting (3.73)–(3.74) into the field equations, a little algebra yields

\[
r^2 \frac{d^2 K(r)}{dr^2} - K(r)(K^2(r) - 1) + H^2(r)K(r)
\]

(3.77)

and

\[
r^2 \frac{d^2 H(r)}{dr^2} = 2H(r)K^2(r) + 4H(r) \left( \frac{H^2(r)}{r^2} - r^2 F^2 \right)
\]

(3.78)

where

\[
K(r) = 1 - gr W(r) \quad \text{and} \quad H(r) = gr F(r).
\]

(3.79)

This is a set of coupled 'non-autonomous' differential equations, in technical parlance. Although much simpler than the parent field equations, these are still not easy to solve. However, in the limit \( \lambda \to 0 \), with \( g \) and \( F \) fixed, it has been shown (Prasad and Sommerfield 1975, Bogomol’nyi 1976) that the equations (3.77)–(3.78) are solved by the particularly simple functions

\[
K(r) = \frac{r F}{\sinh(rF)} \quad \text{and} \quad H(r) = \frac{r F}{tanh(rF)} - 1
\]

(3.80)

Note that although the last term in the Lagrangian (3.42) vanishes when \( \lambda = 0 \), its memory lingers in the solution through the boundary conditions (3.57) and (3.75). The functions (3.80) can be inserted into the ansatz (3.79), (3.73) and (3.74), to obtain an exact static solution in the \( \lambda \to 0 \) limit, belonging to the \( Q = 1 \) sector and carrying a magnetic charge \( 1/r \).

In the limit \( \lambda \to 0 \), our model enjoys other nice features besides yielding the simple analytic solution (3.80) for the \( \text{an} \)-monopole. In particular Bogomol’nyi (1976) has derived an inequality relating the energy of a static configuration to its topological index, very similar to the result (3.31) for the \( O(3) \) model. When \( \lambda = 0 \), the energy given in (3.52) reduces for a static solution with \( A_{1r} = 0 \), to

\[
E = \int d^4x \left[ \frac{1}{4} G^a_2 G_2^a + \frac{1}{2} D_1 \phi^r D_1 \phi^r \right]
\]

\[
= \frac{1}{4} \int d^4x \sum_{a \neq 0} (g^a - \epsilon_{a2} D_1 \phi^r)^2 + \frac{1}{4} \int d^4x \sum_{a \neq 0} g^a_2 D_1 \phi^r .
\]

(3.81)

The second term may be written as

\[
\int d^4x \frac{1}{4} \epsilon_{a2} g^a_2 D_1 \phi^r = \frac{1}{4} \int d^4x \sum_{a \neq 0} (g^a - \epsilon_{a2} D_1 \phi^r)
\]

(3.82)

if we use the identity \( D_1 \phi^a \to 0 \) where \( \phi^a \) is the dual field \( \phi^{a} \rightarrow G_2^k \). This identity follows from (3.43)–(3.44) (see the derivation of the Eucibian analogue (4.20) in chapter 4).

Now, consider the gauge-invariant electromagnetic tensor \( F_{\mu \nu} \) in (3.67). As \( r \to \infty \), \( D_i \phi^r \to 0 \) and \( D^a \phi^r \to 0 \) for any finite-energy configuration. Hence the magnetic field obtained from (3.67) becomes, asymptotically,

\[
B_i = \frac{1}{4} \epsilon_{a2} F_{ij} \to (1/2) \epsilon_{a2} G^a_2 \phi^r.
\]

Using this, we can write (3.82) as

\[
F_i \cdot dB_i = 4 \pi m F = 4 \pi (Q/r) F
\]

(3.83)
where \( w \) is the monopole charge, related to the homotopy index \( Q \) by (3.74). Thus the energy in (3.81) can be written as

\[
E = \frac{4\pi Q F}{g} + \int d^2x \sum_{i=1}^3 \left( G_i - \epsilon_{ijk} D_k \phi \right)^2 \geq \frac{4\pi Q F}{g}.
\]

(3.83)

In any given \( Q \)-sector, the energy is clearly minimised if and only if the fields satisfy the 'Bogomol'nyi condition:

\[
G_i = \epsilon_{ijk} D_k \phi.
\]

(3.84)

These results are clearly similar to eq. (3.31)–(3.32) obtained for the O(3) model. If a field configuration satisfies the Bogomol'nyi condition (3.84), then it minimises the static energy in that \( Q \)-sector and is therefore a static classical solution in that sector. In particular, for \( Q = 1 \), it may be verified that the Prasad–Sommerfield solution (3.80) when inserted into the ansatz (3.73)–(3.74) does satisfy (3.84). Accordingly, the equality in (3.83) tells us that the monopole will have a mass \( 4\pi F / g \). Notice that like the kink solution in chapter 2, the monopole is heavy when the coupling constant \( g \) is small.

For \( Q > 1 \) or \( \lambda \neq 0 \), no analytical solutions are available so far. The numerical work and suggestive arguments given by 't Hooft and Polyakov in the original papers certainly indicate that a non-singular \( Q = 1 \) solution exists even when \( \lambda \neq 0 \), with some finite energy. A more rigorous study of the existence of such solutions has been carried out by Tuyn in et al. (1976).

Apart from all its other interesting properties, the monopole solution is, at the very least, a solitary-wave solution of the model (3.42), which is a fairly complicated coupled field system in \( 3 + 1 \) dimensions. There is no reason, however, to believe that it is a soliton in the strict sense defined in chapter 2.

3.5. More on monopoles and dyons

In the preceding section we presented the homotopy considerations that led to multi-monopole solutions of the model (3.42). We also discussed the single-monopole solution in some detail. Subsequent to these developments a great deal more work has been done in the general area of monopole-like solutions, both for the model (3.42) and for its generalisations. Space does not permit us to discuss all that work in the same detail as we discussed the basic monopole solution. We shall be content with making a collective mention of some of these developments, along with a few qualitative remarks and references. (For more discussion and references see the reviews by Marciano and Pagels (1978), and Actor (1979).)

Soon after the single-monopole solution of the model (3.42) was proposed, Julia and Zee (1975) pointed out that the same model also yields dyons, i.e., objects carrying both electric and magnetic charge. Note that the monopole solution described by the ansatz (3.73)–(3.74) carries no net electrical charge. Since the fields are time-independent, and further \( A_0 = 0 \), the electric field

\[
E_i = -F_{i0} = 0
\]

where \( F_{\mu\nu} \) is the gauge-invariant electromagnetic tensor (3.68). Julia and Zee proposed solutions which are still static, but have non-zero \( A_0 \). They generalised the ansatz to include

\[
A_0 = \lambda J(\tau) / \sqrt{r^2}
\]

with the boundary condition \( J(\tau) \to 0 \) as \( r \to 0 \). The fields \( A_0 \) and \( A_0 \) continue to have the form (3.73)–(3.74). In the place of (3.77)–(3.78), the equations obeyed by \( J(\tau), H(\tau) \) and \( K(\tau) \) are

\[
\begin{align*}
\tau^2 d^2K / dr^2 & = K(K^2 - J^2 + J^2 - 1) \\
\tau^2 d^2H / dr^2 & = 2H(K^2 + J(\tau^2 / 4 + r^2 - r^2 F^2)) \\
\tau^2 d^2J / dr^2 & = 2J K^2.
\end{align*}
\]

In terms of these functions, the electric charge is

\[
q = \frac{4\pi}{g} \int d^2x \left( \text{div} F \right) = \frac{8\pi}{g} \int_0^\infty \frac{JK^2}{r} dr.
\]

(3.85)

In the \( \lambda \to 0 \) limit, exact solutions can again be found (Prasad and Sommerfield 1975, Bogomol'nyi 1976). These are

\[
K(\tau) = \frac{\tau^2 F}{\sinh(\tau^2 / 4 F)} \quad \text{and} \quad H(\tau) = \cosh(\tau^2 / 4 F)(\sinh(\tau^2 / 4 F) - 1)
\]

(3.86)