where $\mu(n)$ is the Möbius function defined as follows:

$$\mu(n) = \begin{cases} 
0 & \text{if } n \text{ is divisible by a prime square} \\
1 & \text{if } n \text{ is a product of an even number of distinct primes} \\
-1 & \text{if } n \text{ is a product of an odd number of distinct primes}
\end{cases} \quad (1.9)$$

For instance, $\mu(2) = \mu(3) = \mu(5) = \mu(7) = -1$; $\mu(6) = \mu(2 \cdot 3) = 1$; $\mu(4) = \mu(2^2) = 0$.

$R(x)$ is an entire function of $\ln x$ with the following expansion

$$R(x) = 1 + \sum_{n \geq 1} \frac{1}{n \zeta(n+1)} \frac{(\ln x)^n}{n!}, \quad (1.10)$$

where $\zeta$ is the Riemann zeta function of $s = \sigma + i \tau$, which is defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \geq 2} (1 - p^{-s})^{-1} \quad (1.11)$$

for $\sigma > 1$ and by analytic continuation for $\sigma \leq 1$, $s \neq 1$. Notice the connection between $\zeta(s)$ and the prime numbers $p$. For $x = 10^6$ one has $\pi(10^6) = 78498$ and the first two terms of (1.8) give $\text{Li}(10^6) - \frac{1}{2} \text{Li}(10^3) = 78628 - \frac{1}{2} \times 178 = 78539$. On Fig.1.2 is shown the difference $R(x) - \pi(x)$. As can be seen, no structure is anymore present. It seems now that one can consider $R(x)$ to be the smooth behaviour of $\pi(x)$.

Fig.1.3 - Plot of $R(x)$ and $x/\ln x$ for $x \leq 5 \times 10^4$ (taken from Ref.[2a-77])

after subtracting $R$ to $\pi$, only fluctuations are left out. In fact, Riemann, although unable to prove the PNT\(^{*}\), did something even more astonishing. He derived an exact relation for $\pi(x)$:

$$\pi(x) = R(x) - \sum_{\rho} R(x^\rho) \quad (1.12)$$

\(^{*}\) The PNT was proved simultaneously and independently by Hadamard and de la Vallée-Poussin in 1896. Hadamard was born in 1865 and de la Vallée-Poussin in 1866 and they died in 1963 and 1962 respectively!