The quality of the approximation $\pi(x) \approx \text{Li}(x)$ is, for many purposes, very good. For instance, for $x \leq 10^7$, the relative error $|\pi(x) - \text{Li}(x)|/\pi(x)$ is smaller than $5 \times 10^{-5}$. On Fig.1.2 is plotted, for $x \leq 10^7$, the difference $\text{Li}(x) - \pi(x)$. It can be seen that although this difference is small, it is not featureless; for instance, for $x \leq 10^7$, it steadily increases.

![Fig.1.2 - Difference between Gauss (Riemann) approximation $\text{Li}(x)$ (R(x)) to $\pi(x)$ and $\pi(x)$ for $x \leq 10^7$ (taken from Ref.[Za-77])](image)

What about rigorous, "non-empirical", results? One of the main questions, in the middle of the 19th century, was to prove the prime number theorem (PNT) namely

$$\pi(x) \sim \frac{x}{\ln x} \quad (1-5)$$

Notice that Gauss approximation is consistent with the PNT. The first major result in the direction of the PNT was obtained by Chebyshev in 1850, who proved that

$$0.9 \frac{x}{\ln x} < \pi(x) < 1.1 \frac{x}{\ln x} \quad (1-6)$$

for sufficiently large $x$. Although the PNT is true, the approximation

$$\pi(x) \sim \frac{x}{\ln x} \quad (1-7)$$

is much poorer than the approximation introduced by Gauss (see Fig.I.3).

Riemann, in his famous memoir "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" introduced, based on empirical evidence and intuition, a better approximation $R(x)$ to $\pi(x)$

$$\pi(x) \sim R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{Li} \left( x^{\frac{1}{n}} \right), \quad (1-8)$$