Homework Set # 5, due March 24, 2008

1. Consider the classical field theory

\[ L = \frac{1}{2} \sum_{\mu=1}^{2} \sum_{a=1}^{3} (\partial_{\mu} \phi_{a})^2, \quad \text{with} \quad \sum_{a} = 1^{3} \phi_{a}^2 = 1. \quad (1) \]

a) Given that \( \Pi_s(S_2) = Z \) argue that this theory has topological solutions of finite energy.
   • Show that
   \[ Q = \frac{1}{8\pi} \int \epsilon_{\mu\nu} \epsilon_{abc} \partial_{\mu} \phi_{a} \partial_{\nu} \phi_{b} \phi_{c} d^2x \quad (2) \]
   is a topological invariant, i.e. it is insensitive to small deformations of \( \vec{\phi} \). \( Q \) is the topological charge of the solution.

b) Show that the energy of the finite energy solution satisfies

\[ E \geq 4\pi Q. \quad (3) \]

2. Show that for the hedgehog ansatz, the fourth order term in the Skyrme model is given by

\[ M_4 = 16\pi e^2 \int_{0}^{\infty} r^2 dr \frac{\sin^2 F}{r^2} \left( \frac{\sin^2 F}{r^2} + 2(\partial_r F)^2 \right). \quad (4) \]

3. In electrodynamics the scalar field produced by an electron at the origin satisfies the Poisson equation

\[ (\partial_\mu)^2 \phi(\vec{r}) = -4\pi e \delta(\vec{r}). \quad (5) \]

a) Show that the radial dependence of the field is given by

\[ \phi(r) = \frac{e}{r}. \quad (6) \]

b) For a nucleon, the scalar field satisfies the Klein-Gordon equation

\[ ((\partial_\mu)^2 - \frac{1}{r_0^2}) \phi(\vec{r}) = -4\pi e \delta(\vec{r}). \quad (7) \]

Show that the radial dependence of \( \phi(r) \) is given by

\[ \phi(r) = -g \frac{e^{-r/r_0}}{r}. \quad (8) \]