\[ L = \left[ p, \omega \right] = \left( \frac{\partial L}{\partial \dot{q}} \right) \left( -\frac{\partial L}{\partial \dot{p}} \right) - \left( \frac{\partial L}{\partial q} \right) \left( \frac{\partial L}{\partial \dot{p}} \right) \]

\[ = \frac{\partial^2 L}{\partial p \partial \dot{q}} \frac{\partial q}{\partial \dot{p}} = \frac{\partial^2 L}{\partial q \partial \dot{p}} = \begin{cases} 1 & \text{symmetric} \\ x & \text{anti-symmetric} \end{cases} \]

To prove the general case, we first show that Lagrange brackets are invariant under canonical transformations.

Lagrange brackets are defined as

\[ \{ \mathbf{u}, \mathbf{v} \} = \sum \left( \frac{\partial q_i}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \dot{q}_i} - \frac{\partial q_i}{\partial \dot{p}_j} \frac{\partial q_j}{\partial \dot{p}_i} \right) \]

We now use the generating function

\[ F(q, p) \]

then

\[ p_i = \frac{\partial F}{\partial \dot{q}_i}, \quad q_i = \frac{\partial F}{\partial \dot{p}_i} \]

and

\[ \frac{\partial q_i}{\partial \dot{p}_j} = \frac{\partial }{\partial \dot{p}_j} \left( \frac{\partial F}{\partial \dot{q}_i} \right) = \frac{\partial F}{\partial \dot{q}_i} \frac{\partial q_j}{\partial \dot{p}_i} + \frac{\partial F}{\partial q_j} \frac{\partial q_j}{\partial \dot{p}_i} \]

\[ \text{same for } \frac{\partial q_i}{\partial \dot{p}_j} \]

\[ \{ \mathbf{u}, \mathbf{v} \} = \frac{\partial F}{\partial \dot{q}_i} \left( \frac{\partial q_i}{\partial \dot{p}_j} \frac{\partial q_j}{\partial \dot{p}_i} - \frac{\partial q_i}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \dot{p}_i} \right) + \frac{\partial F}{\partial \dot{q}_i} \left( \frac{\partial q_i}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \dot{p}_i} - \frac{\partial q_i}{\partial \dot{p}_j} \frac{\partial q_j}{\partial \dot{p}_i} \right) \]

\[ + \frac{\partial F}{\partial \dot{p}_i} \left( \frac{\partial q_i}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \dot{p}_i} - \frac{\partial q_i}{\partial \dot{p}_j} \frac{\partial q_j}{\partial \dot{p}_i} \right) \]

\[ \text{add this (it is 0)} \]