6.b) Perturbation theory for the anharmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 + \epsilon x^3 \]

\[ m = 1 \]

\[ x'' + \omega_0^2 x + \epsilon x^3 = 0 \]

Perturbative expansion:

\[ x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \]

\[ x_0'' + \omega_0^2 x_0 = 0 \]

\[ x_1'' + \omega_0^2 x_1 + \omega_0^2 (x_0 + \epsilon x_1 + \epsilon^2 x_2) + \epsilon x_0^3 = 0 \]

\[ \mathcal{O}(\epsilon^0) \]

\[ x_0 = -\omega_0^2 x_0 \]

\[ \mathcal{O}(\epsilon) \]

\[ x_1'' + \omega_0^2 x_1 + x_0^3 = 0 \]

Initial conditions:

\[ x(0) = a \quad x'(0) = 0 \quad x''(0) = 0 \quad x_{\epsilon k}(0) = 0 \]

\[ x_0(0) = a \]

\[ x_0'(0) = 0 \]

\[ x_0''(0) = 0 \]

\[ x_1'' + \omega_0^2 x_1 = -a^3 \cos^3 \omega_0 t = -\frac{3}{4} a^3 \cos \omega_0 t - \frac{a^3}{3} \sin^3 \omega_0 t \]

Solution:

\[ x(x) = \frac{-a^3}{Pr^2} (3 \omega_0 t \sin^2 \omega_0 t + \frac{1}{3} \cos 3 \omega_0 t) \]

Diverges for \( t \to \infty \)

Reason for divergence: true resonance frequency is \( \omega_0 + \omega_0 \)