Small oscillations

Small oscillations about the equilibrium point are important in many applications. E.g. this gives the resonance frequencies of a bridge.

Equilibrium point \( q_{0i} \)

\[ x_i = q_i - q_{0i} \]

At equilibrium \( \frac{\partial U}{\partial q_i} \mid_{q_i = q_{0i}} = 0 \)

\[ U = U_0 + \frac{1}{2} \sum_{ij} K_{ij} x_i x_j + \ldots \quad \text{neglect} \]

Kinetic energy \( T = \frac{1}{2} \sum_{ij} (q_i - q_{0i}) \dot{q}_i \dot{q}_j \)

![Image with arrows pointing to second order terms in \( x_k \)]

In \( x_k \) we can take \( \dot{q}_k = \text{constant} \equiv m_{ij} \)

\[ m_{ij} = m_{ij} \]

\[ m_{ij} > 0 \quad \text{because kinetic energy is positive} \]

Lagrangian

\[ L = \pm \sum_{ij} m_{ij} x_i \dot{x}_j - K_{ij} x_i x_j \]

\[ E_L \quad m_{ij} \ddot{x}_j - K_{ij} x_j = 0 \]

This is a differential equation with constant coefficients and can be solved by substituting \( x_k = A_k e^{i\omega_k} \)