The brachistochrone

What is the shape of a path that minimizes the time to go from 1 to 2 starting from rest from 1 in a gravitational field with acceleration g?

\[ \frac{1}{2} m v^2 = (f(o) - f(x)) mg \]

\[ f(o) - f(x) \geq 0 \]

otherwise \( m \) never gets to \( x \) to \( 2 \)

\[ v^2 = 2(f(o) - f(x)) g \]

The time it takes is

\[ T = \int \frac{ds}{v} = \int \frac{dx \sqrt{1 + f'^2(x)}}{\sqrt{v^2 - g^2 f(x)}} \]

Minimizing \( T \) \( \Rightarrow \) E-L eqs

\[ \frac{d}{dx} \left( \frac{\partial F}{\partial f'} \right) - \frac{\partial F}{\partial f} = 0 \]

\[ \frac{1}{\sqrt{1 + \left( \frac{f'}{f} \right)^2}} \left( \frac{f'}{f} \right)^{3/2} - \frac{\sqrt{1 + \left( \frac{f'}{f} \right)^2} (f' - f)^{1/2}}{(f(o) - f)^{3/2}} = 0 \]