These lecture notes are based on the text book “Mechanics” by Landau and Lifshitz, which is the required textbook for this course.
Lecture 1: Historical perspective

1a) **Plato**: 429-348 BC - ideas are central and are not tested by observations

**Aristotle**: 387-322 BC - a force is needed to move an object
- heavenly objects are lighter because light objects move up

**Euclid**: 300 BC, geometry

**Archimedes**: 287-212 BC, statics, recursive reasoning

In the western world not much did happen until the 13th century. However, mathematics was developed in India and in the Muslim world (Alexandria).

One reason for lack of progress is the use of Roman numerals.

**Ptolemy**: 100-170 epicycles

**Neoplatonism**: Platoism is adopted by Christianity
- Sun and planets move on spheres, in modern language, on a 2-brane.
Because of some mysterious force, they stay on spheres. Like a brain in string theory onto which a particle is confined.

Thomas Aquinas: 1225-1274; he tried to create a worldview that explains nature.

Fibonacci: 1180-1250, introduced algebra and arithmetics from North-Africa.

Copernicus: 1473-1543, sun centered solar system, demise of religious authority.

Tycho Brahe: 1546-1601, astronomical observations (orbit of Mars). Because of its large eccentricity, it helped Kepler to discover his laws.

Simon Stevin: 1548-1620, introduced decimal numbers; important for quantitative measurements.
Descartes 1596-1650, Introduced equations into physics

Galileo 1564-1642, Introduced inertia; if no force acts on a body, its velocity remains constant

Kepler 1571-1630, Kepler’s laws

1) A planet has an elliptical orbit with the sun in one of its focal centers.

2) A line from a planet to the sun covers equal areas in equal times.

3) The period of an orbit is proportional to $a^{3/2}$ (where $a$ is the long axis of the orbit).

Newton 1642-1727, Newton’s laws explain Kepler’s laws.

1) First law – $\vec{v}$ is constant if no forces act on a body.

2) Second law \[
\frac{d\vec{p}}{dt} = \vec{F}
\]
3) \textbf{Third Law} \quad \vec{F}_{ix} = -\frac{2}{F_{ki}}

16) 2) \textbf{Review of Classical Mechanics}

\[ \dot{q} = \frac{d^2 q}{dt^2} \quad \dot{p} = \frac{d^2 p}{dt^2} \]

\[ \vec{F} = \frac{d\vec{p}}{dt} \]

\[ \vec{F} = 0 \quad \Rightarrow \quad \vec{p} = \text{constant} \]

\textbf{Angular Momentum} \quad \vec{L} = \vec{r} \times \vec{p}

\textbf{Torque} \quad \vec{N} = \vec{r} \times \vec{F}

\[ \frac{d}{dt} \left( \frac{d\vec{L}}{dt} \right) = \frac{d^2 \vec{L}}{dt^2} \]

\[ \vec{L} = \vec{r} \times \vec{F} = \vec{N} \]

\[ \vec{N} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant} \]

\textbf{Work} \quad W_{12} = \sum \int F \cdot ds

\[ = \int m \frac{dv}{dt} \cdot \vec{v} \cdot dt = \frac{1}{2} \int m \vec{v} \cdot d\vec{v} \]

\[ = \frac{1}{2} (m v_e^2 - m v_i^2) \]

\[ \Rightarrow \quad W_{1e} = \frac{1}{2} (v_e^2 - v_i^2) \]
Conservative Force

\[ F \cdot ds = 0 \Rightarrow \nabla \times F = 0 \]

= Work is path independent

\[ \nabla \cdot F = 0 \Rightarrow F = -\nabla U \]

then \[ W_{12} = - \int_{1}^{2} \nabla U \cdot ds = U_{2} - U_{1} \]

= \[ T_{f} + U_{f} = T_{e} + U_{e} \]

Many particles

\[ F_{12} = - F_{21} \]

\[ F = F_{e} + \sum_{i \neq j} F_{ji} \]

= \[ \sum_{i} F_{ei} = \sum_{i} F_{ei}^{2} + \sum_{i \neq j} F_{ji} \]

\[ \text{total momentum} = 0 \text{ 3rd law} \]

\[ \text{total momentum is conserved if external force is zero} \]
Lecture #2

\[ \frac{d^2 \vec{r}}{dt^2} = \vec{F} \]

\[ \vec{F}_{\text{ext}} = -\nabla \vec{V} \]

\[ \vec{L} = \vec{r} \times \vec{p} \]

\[ \vec{L} = \vec{N} \]

\[ W = \int \vec{F} \cdot d\vec{s} \]

\[ W_{\text{ext}} = T_e - T_f \]

Conservative

\[ \nabla \times \vec{F} = 0 \]

\[ \oint \vec{F} \cdot d\vec{s} = 0 \]

\[ \vec{F} = -\nabla \vec{V} \]

Today

System of particles

1a) Generalized coordinates and constraints

1b) Constrained forces

2a) Lagrange equations
total angular momentum

\[ L = \sum_i r_i \times p_i \]

\[ \frac{dL}{dt} = \sum_i \frac{d}{dt} (r_i \times p_i) + \sum_i r_i \times \frac{dp_i}{dt} \]

\[ = \sum_i r_i \times \left( \frac{\partial}{\partial t} \frac{\vec{r}}{t} + \sum_{j \neq i} \frac{\vec{F}_{ji}}{t} \right) \]

\[ = \sum_i \vec{r}_i \times \frac{\vec{F}_i}{t} + \sum_i (\vec{r}_i - \vec{R}) \times \vec{F}_{ji} \]

\[ \text{vector} \parallel \vec{F}_{ji} \]

\[ \Rightarrow L \text{ is constant if external torque is zero.} \]

Center of mass \[ \vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} \]

\[ \vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times \vec{p}_i + \sum_i (\vec{r}_i - \vec{R}) \times \vec{F}_{ji} \]

\[ = \vec{R} \times \vec{p} + \sum_i (\vec{r}_i - \vec{R}) \times \vec{F}_{ji} \]

\[ = \vec{R} \times \vec{p} + \sum_i (\vec{r}_i - \vec{R}) \times (\vec{F}_{ji} - \vec{F}_i) \]

(because \( \sum_i (\vec{r}_i - \vec{R}) = \vec{0} \))
Energy of a system of particles

\[ T = \frac{1}{2} \sum_i m_i \dot{r}_i^2 \]

Kinetic energy

\[ V = V(r_i - r_j) \]

Potential energy

Typically, \[ V = V(r_i - r_j) \]

\[ W = T + V \]

\[ \text{cm velocity} \quad \dot{V}_{cm} = \frac{\sum m_i \dot{r}_i}{\sum m_i} \]

\[ T = \frac{1}{2} \sum m_i (\dot{r}_i - \dot{V}_{cm})^2 \]

\[ + \frac{1}{2} \cdot 2 \sum \dot{V}_{cm} m_i (\dot{r}_i - \dot{V}_{cm}) \]

\[ \frac{1}{2} M_{cm} \dot{V}_{cm}^2 \]

\[ \text{internal kinetic energy} = 0 \]
1c) Generalized coordinates and constraints

\[ N \text{ particles } \Rightarrow 3N \text{ degrees of freedom} \]

but we can have constraints

a particle has to move on a circle
or in a plane

a constraint is a relation between
the coordinates and velocities of
the particles

\[
\text{holonomic constraint:} \\
\mathbf{f}_k(\mathbf{r}, \mathbf{\dot{r}}, \mathbf{\ddot{r}}, t) = 0, \quad k = 1, \ldots, p
\]

Instead of the original coordinates
we use generalized coordinates
\[ \mathbf{q}_1, \ldots, \mathbf{q}_{3N-k} \]

Example:

\[ x_1^2 + x_2^2 = 1 \]

\( q \) is generalized coordinate
with \( x_1 = \cos q, \quad x_2 = \sin q \)

\[
\text{non-holonomic constraint:} \\
\mathbf{f}_k(\mathbf{r}, \mathbf{\dot{r}}, \mathbf{\ddot{r}}, \mathbf{\dot{q}}, \mathbf{\ddot{q}}, t) = 0, \quad k = 1, \ldots, p
\]
Example

Disk that can only move in plane
No slipping $\Rightarrow \dot{\theta} = r \dot{\phi}$
Coordinates $x, y, \theta, \phi$

$\dot{x} = r \cos \theta \dot{\phi}$
$\dot{y} = r \sin \theta \dot{\phi}$

\[ dx - r \sin \theta d\phi = 0 \]
\[ dy + r \cos \theta d\phi = 0 \]

Cannot be used to eliminate coordinates $\Rightarrow$ these are nonholonomic constraints.

1d) Constrained forces

Surface

Assume that we have holonomic constraint

$F_x(x) = 0$

$F_x(x + \Delta x) = F_x(x) = 0$

$F_x(x) + \Delta x \cdot \nabla F_x(x)$

to cancel this term we need a force $\dot{c}_x = \nabla F_x$
\( \ddot{x}_k \) are the constrained forces

**Newton equation**

\[
\sum \frac{\partial V}{\partial x_k} \dot{x}_k + \sum \frac{\partial F}{\partial x_k} = m \ddot{x}_k
\]

3N + p unknowns \( x_k, \lambda_p \)

3N + p eqs:

\[
f_k = 0, \quad f_p = 0
\]

Introduce generalized coordinates

\[
q_i \rightarrow \phi_i \quad x_i = x_i(q_1, \ldots, q_m)
\]

then the constraints are satisfied automatically.

1) Lagrange Equations

We are now going to rewrite the Newton eqs in terms of the \( q_i \).

We multiply Newton eq by \( \frac{\partial x_i}{\partial q_k} \)

\[
\sum \frac{\partial V}{\partial x_i} \dot{x}_i = -\frac{\partial f_k}{\partial q_k} \frac{\partial x_i}{\partial q_k}
\]

So now we have exactly 3N - p newton equation
Lecture #3

n particles \[ \sum \dot{p}_i = \sum F_i \]

\[ \begin{align*}
K & = \frac{1}{2} \sum m_i \dot{r}_i^2 \\
T & = \frac{1}{2} \sum m_i (\dot{r}_i - \dot{r}_m)^2 + \frac{1}{2} M \dot{u}_m^2
\end{align*} \]

holonomic constraint

non-holonomic constraint

Today

1a) Constant forces
1c) Lagrange equation
1f) Example
1g) Galilean invariance
\[ \sum_{\nu} \mathbb{V} \frac{\partial x_{\nu}}{\partial q^e} = \sum_{\nu} \mathbb{V} \frac{\partial x_{\nu}}{\partial q^e} - 2 \mathbb{V} \]

\[ \sum_{\nu} m \frac{\partial \mathbb{V}}{\partial q^e} = \frac{d}{dt} \sum_{\nu} m \frac{\partial \mathbb{V}}{\partial q^e} - \frac{d}{dt} \sum_{\nu} m \frac{\partial \mathbb{V}}{\partial q^e} \]

\[ \frac{d \mathbb{V}}{dt} = \sum_{\nu} \frac{\partial \mathbb{V}}{\partial q^e} \frac{d q^e}{dt} + \frac{\partial \mathbb{V}}{\partial q^e} \]

\[ \Rightarrow \frac{d \mathbb{V}}{dt} = \frac{\partial \mathbb{V}}{\partial q^e} \]

\[ \Rightarrow \frac{d \mathbb{V}}{dt} = \frac{\partial \mathbb{V}}{\partial q^e} \]

\[ = \sum_{\nu} m \frac{\partial \mathbb{V}}{\partial q^e} = \frac{d}{dt} \sum_{\nu} m \frac{\partial \mathbb{V}}{\partial q^e} = \sum_{\nu} \frac{d^2}{dt^2} m \frac{\partial \mathbb{V}}{\partial q^e} \]

\[ = \frac{d}{dt} \left[ \sum_{\nu} \frac{1}{2} m \frac{\partial \mathbb{V}}{\partial q^e} \right] - \frac{1}{2} m \frac{\partial \mathbb{V}}{\partial q^e} \]

Kinetic energy \( T = \sum_{\nu} \frac{1}{2} m \mathbb{V}^2 \)

\[ \Rightarrow \frac{d}{dt} \frac{\partial T}{\partial q^e} = \frac{\partial}{\partial q^e} \left( T - V \right) \]

\[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}^e} (T-V) \quad \text{if } V \text{ does not depend on } q_i \]

Lagrangian \( L = T - V \)

Lagrangian Equations \( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^e} = \frac{\partial L}{\partial q^e} \)
$L$ is not unique

\[ L \Rightarrow \frac{\partial F}{\partial t} + \frac{\partial F}{\partial h} \frac{\partial h}{\partial t} = L + \frac{\partial F}{\partial h} \frac{\partial h}{\partial s} + \frac{\partial F}{\partial s} \frac{\partial s}{\partial t} \]

\[ \frac{\partial}{\partial t} \frac{\partial (L - L')} {\partial x} = \frac{\partial \partial F}{\partial x} \frac{\partial x}{\partial t} \frac{\partial \partial F}{\partial x} \frac{\partial x}{\partial t} \]

\[ \frac{\partial}{\partial x} \frac{\partial (L - L')} {\partial x} = \frac{\partial \partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial \partial F}{\partial x} \frac{\partial x}{\partial t} \]

\[ \frac{\partial}{\partial x} \frac{\partial (L - L')} {\partial x} = \frac{\partial \partial F}{\partial x} \frac{\partial x}{\partial t} \]
Example of a Lagrangian

Double pendulum

\[ T_1 = \frac{1}{2} m_1 \dot{\phi}_1^2 \]

\[ T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \]

\[ x_2 = r_1 \cos \phi_1 + l_2 \cos \phi_2 \]

\[ y_2 = r_1 \sin \phi_1 + l_2 \sin \phi_2 \]

\[ -T_2 = \frac{1}{2} m_2 (r_1 \dot{\phi}_1^2 + l_2 \dot{\phi}_2^2 + 2r_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2) \]

Potential energy:

\[ U_1 = -m_1 g r_1 \cos \phi_1 \]

\[ U_2 = -m_2 g l_2 \cos \phi_2 \]

\[ L = T_1 + T_2 - U_1 - U_2 \]
Galileo's relativity principle

It is always possible to find coordinates such that time and space are homogeneous and space is isotropic. Such frame is called an inertial frame. This implies that a free particle has the same equations of motion all over space-time.

\[ L_{\text{free}} = L (\dot{u}^2) \]

Lagrangian of free particle does not depend on \( q, \) and \( t. \) Because space is isotropic it can only depend on

\[ u^2 = \sum \dot{q}_k^2 \]

Lagrangian equation of motion \( \frac{\partial}{\partial \dot{q}^i} \frac{\partial L}{\partial q^i} = 0 \)

\[ \frac{\partial L}{\partial \dot{q}^i} = \text{constant} \Rightarrow \dot{u} = \text{constant} \]

Law of inertia \( \dot{u} \) is constant for free motion.

Equations of motion are the same in any inertial frame. Inertial frames are related by Galilean transformation

\[ x = x' + vt, \]
\[ t = t'. \]
Lagrangian for a free particle

\[ \dot{x}^2 + \dot{\omega}^2 \Rightarrow \dot{\xi} = \dot{x} + \dot{\omega} \]

Lagrangian should be invariant under Galilean transformations

If

\[ L(\xi, \dot{\xi}) = \sum a_k (\dot{\xi}^2)^k \]

then

\[ L(\dot{\xi}, \dot{\xi}) = \sum a_k (\dot{\xi} - \dot{x} - \dot{\omega} \cdot \dot{\xi} \cdot \dot{\omega})^k \]

\[ = \sum a_k (\dot{\xi}^2 + \dot{\omega}^2 - 2 \dot{x} \cdot \dot{\omega} \cdot \dot{\xi}) \]

New and old Lagrangians differ by

For \( k = 1 \)

\[ \frac{3}{2} \dot{\xi}^2 = \frac{1}{2} (\dot{\xi}^2 - 2 \dot{x} \cdot \dot{\omega} \cdot \dot{\xi}) \]

For \( k = 1 \)

\[ \frac{3}{2} \dot{\xi}^2 (\dot{\xi}^2 - 2 \dot{x} \cdot \dot{\omega} \cdot \dot{\xi}) + (\dot{\xi}^2 - 2 \dot{x} \cdot \dot{\omega} \cdot \dot{\xi}) \]

\[ = \frac{\dot{x}^4 - 4 \dot{x} \cdot \dot{\omega} \cdot (\dot{\xi}^2 - 2 \dot{x} \cdot \dot{\omega} \cdot \dot{\xi}) + 4 (\dot{\xi} \cdot \dot{\omega} \cdot \dot{\xi})}{ \frac{1}{2} \dot{\omega}^2 - \dot{x} \cdot \dot{\omega} \cdot \dot{\xi} } \]

Cannot be written as total derivative by for \( \xi, \dot{\xi} \) need combination \( \dot{\xi} \left( \frac{\partial}{\partial \xi} \right) \) \[ \frac{\partial}{\partial \xi} \left( \frac{\dot{\omega}^2}{2} \right) \]

but this also gives a term \( \frac{\partial \dot{\omega}^2}{\partial \xi} \cdot \dot{\xi} \)

\[ \Rightarrow \text{Lagrangian can only contain the } k = 1 \text{ term} \]

Notice that if two Lagrangians differ by a total derivative, the equations of motion are the same.
\[ L = \frac{1}{2} m \dot{u}^2 \]

constant which we define as the mass

System of free particles \[ L = \sum \frac{1}{2} m \dot{u}_i^2 \]

Lagrangian of free particle in polar coordinates

\[ d\tau = (\frac{dr}{\dot{r}})^2 + r^2 d\phi^2 \]

\[ L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 \]

Lagrangian of free particle in spherical coordinates

\[ ds^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2 \]

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \]
1. \textbf{Hooke's machine}

\[ T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_2 \dot{x}_2^2 \]

\[ U = -m_1 g x - m_2 g (L - x) \]

\[ L = T - U \]

\[ \text{Lagrangian eqs:} \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = \ddot{x} = \frac{(m_2 - m_1) g}{m_1 m_2} \]

2. \textbf{Variational formulation of Lagrange q1}

\[ L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \]

\[ \text{Action} \quad S = \int_{t_1}^{t_2} L \, dt \]

\[ \text{Hamilton's principle:} \quad \dot{q}_i = \frac{\partial L}{\partial \dot{q}_i} \]

\[ \text{classical motion from} \quad t_1 \text{ to} \quad t_2 \text{ minimizes} \quad S \Rightarrow S = \]

\[ q_n \rightarrow q_n + \delta q_n \]

\[ = \int_{t_1}^{t_2} L \left( q_n + \delta q_n, \dot{q}_n + \delta \dot{q}_n \right) \, dt \]

\[ \approx \delta q_n \frac{\partial L}{\partial q_n} + \sum_i \delta \dot{q}_n \frac{\partial L}{\partial \dot{q}_n} \, dt \]

\[ \text{partial integrate} \]
\[
\begin{align*}
&= 
\sum_k \left( \delta q_k \frac{\partial L}{\partial q_k} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) + \delta q_k \frac{\partial L}{\partial q_k} \
&= 0 \quad \forall \delta q_k
\end{align*}
\]

\[\delta q = 0 \quad \text{at endpoints} \]

\[\frac{\partial}{\partial q} - \frac{\partial}{\partial t} \frac{\partial}{\partial \dot{q}} = 0 \quad \text{Lagrange equations} \]

2.6)

Lagrange eqs with constraints

ho\text{-}lo\text{-}no\text{m}\text{i}c

\[
L \rightarrow L + \sum_{k=1}^{n} \lambda_k \mathbf{f}_k
\]

\[
\Rightarrow \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial}{\partial t} \frac{\partial L}{\partial q_k} = \sum_{\alpha=1}^{p} \mathbf{f}_\alpha \frac{\partial \lambda_\alpha}{\partial q_k} = 0
\]

(generalized force)

Nonholonomic constraint \( \sum \mathbf{a}_k \mathbf{e}_k \cdot \dot{\mathbf{q}}_k = 0 \)

\( k = 1, \ldots, p \)

\( \Rightarrow \mathbf{a}_k, \; k = 1, \ldots, p \) are vectors

1 to constraint planes

\( \Rightarrow \) need to add forces \( \sum_{\alpha=1}^{p} \lambda_\alpha \mathbf{e}_\alpha \)

\[
\Rightarrow \text{Lagrange eqs: } \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial}{\partial t} \frac{\partial L}{\partial q_k} - \sum_{\alpha=1}^{p} \lambda_\alpha \mathbf{e}_\alpha
\]
Example 1) Motion on a circle

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]

Constraint: \( F = x^2 + y^2 - R^2 \)

\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + xF \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow m \ddot{x} = 2 \lambda x \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \Rightarrow m \ddot{y} = 2 \lambda y \]

\[ x^2 + y^2 = R^2 \Rightarrow 2x \ddot{x} + 2y \ddot{y} = 0 \]

\[ = 2 \ddot{x} (x) + 2 \ddot{y} (y) = 0 \]

\[ = m (\ddot{x}^2 + \ddot{y}^2) = -2 \lambda (x^2 + y^2) = -2 \lambda R^2 \]

\[ \lambda = -\frac{m \ddot{\theta}^2}{2R^2} \]

Force: \[ 2\lambda \left( \frac{x}{y} \right) = -\frac{m \ddot{\theta}^2}{R^2} \left( \frac{x}{y} \right) \]

Example 2)

Constraint: |\( r \frac{dy}{dx} = \frac{d\theta}{dt} \)

\[ T = \frac{1}{2} M \ddot{x}^2 + \frac{1}{2} Mr^2 \ddot{\theta}^2 \]

\[ V = Mg (l-x) \sin \theta \]

\[ L = T - V \]
\[ EL \quad x : \quad \frac{d}{dt} \left( \frac{2L}{\gamma} - \frac{2L}{\gamma} \right) - \lambda \alpha x = 0 \]

\[ \Theta : \quad \frac{d}{dt} \left( \frac{2L}{\gamma} - \frac{2L}{\gamma} \right) - \lambda \alpha \Theta = 0 \]

\[ r d\Theta - \alpha x = 0 \]

\[ \alpha T - \alpha x \]

\[ \Rightarrow M \ddot{x} - Mg \sin \alpha + \lambda = 0 \quad \Rightarrow M \ddot{x} - Mg \sin \alpha + Mr \ddot{\Theta} = 0 \]

\[ r \dot{\Theta} = \dot{x} = r \ddot{\Theta} = \ddot{x} \]

\[ 2M \ddot{x} = g \sin \alpha M \]

\[ \Rightarrow \ddot{x} = \frac{g \sin \alpha}{2} \]

\[ \Theta = \frac{g \sin \alpha}{2r} \]
2 d) Integrals of motion

F(q, \dot{q}) that stays constant during time evolution

Special integrals of motion: conserved quantities they are additive

If there are s degrees of freedom there are at most 2s-1 constant of motion

Proof motion is fixed by initial conditions: need s positions and s velocities or momenta at t = 0

Integrals of motion do not depend on t_0 - eliminate to

\Rightarrow 2s-1 constants of motion completely fix the initial conditions

\text{eg } s = 1 \quad \dot{x} = \frac{dx}{dt} = x_0 + \dot{x}_0 t

x(t) = x_0 + \dot{x}_0 t \quad \dot{x}(t) = \dot{x}_0 \quad \text{integral of motion}

not a conserved quantity or integral of motion because it depends on \ t_0
A less trivial example

\[ \ddot{x} = a \]
\[ x = x_0 + u_0 t + \frac{1}{2} a t^2 \]
\[ \dot{x} = u_0 + a t \]

\[ u_0 = \dot{x} - a t \]
\[ x_0 = x + (\dot{x} - a t) t + \frac{1}{2} a t^2 \]
\[ x_0 = x + \dot{x} t - \frac{1}{2} a t^2 \]

at \( t = 0 \)
\[ u_0 = \dot{x}(t_o) \]
\[ x_0 = x(t_o) \]

Eliminate \( t \) for \( t \)

\[ x_0 = x + \dot{x} \left( \frac{\dot{x} - u_0}{a} \right) - \frac{1}{2} a \left( \frac{\dot{x} - u_0}{a} \right)^2 \]
\[ = x + \frac{1}{2 a} \left( 2 x^2 - 2 x^2 u_0 - \dot{x}^2 + 2 x u_0 - u_0^2 \right) \]
\[ = x + \frac{1}{2 a} \left( \dot{x}^2 - u_0^2 \right) \]
\[ \Rightarrow x_0 + \frac{u_0^2}{2 a} = x + \frac{\dot{x}^2}{2 a} \]
2c) **Energy**

Homogeneity of time \(\Rightarrow\) Lagrangian does not explicitly on time

\[\frac{\partial L}{\partial t} = 0 \Rightarrow L(q, \dot{q})\]

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q} = 0\]

**Energy**

\[E = H = -L + q \frac{\partial L}{\partial \dot{q}}\]

**Conservative systems**: mechanical systems

for which the energy is

\[L= T - U\]

\[T\] is quadratic in \(\dot{q}\)

\[\Rightarrow \quad q \frac{\partial L}{\partial \dot{q}} = q \frac{\partial T}{\partial \dot{q}} = 2T\]

\[E = 2T - T + U = T + U\]
\[ \delta L = \sum_{a} L \frac{\delta}{\delta \psi_{a}} = \frac{\delta}{\delta \psi_{\text{total}}} L \]

\[ \delta \sum \frac{\partial L}{\partial \dot{x}_{a}} \dot{x}_{a} = 0 \]

\[ \frac{\delta}{\delta t} \sum \frac{\partial L}{\partial \dot{x}_{a}} = 0 \]

Total momentum

2g) Angular momentum

\[ \delta \vec{p} = \delta \vec{\phi} \times \vec{r} \]

\[ \delta \vec{\omega} = \delta \vec{\phi} \times \vec{v} \]

Isotropy of space: Lagrangian is invariant under rotations

\[ \delta L = \frac{\partial L}{\partial \dot{x}_{a}} \delta \dot{x}_{a} + \frac{\partial L}{\partial \dot{\psi}_{a}} \delta \dot{\psi}_{a} \]

\[ = \frac{\partial L}{\partial \dot{x}_{a}} (\delta \phi \times \vec{r})_{\dot{u}} + \frac{\partial L}{\partial \dot{\psi}_{a}} (\delta \phi \times \vec{v})_{\dot{u}} \]

\[ \dot{\psi}_{a} \]

\[ \vec{p}_{u} \]

\[ \Rightarrow \delta L = \vec{p}_{u} (\delta \phi \times \vec{r})_{\dot{u}} + \vec{p}_{u} (\delta \phi \times \vec{v})_{\dot{u}} \]

\[ = \delta \phi \cdot \frac{\partial}{\partial \psi} (\vec{r} \times \vec{p}) \]

\[ = \delta \phi \cdot \frac{\partial}{\partial \psi} (-\vec{p} \times \vec{r}) \]
\[ T = \dot{x} \dot{y} \]

\[
\Rightarrow \frac{\partial T}{\partial \dot{x}} + \frac{\partial T}{\partial \dot{y}} = \ddot{x} \dot{y} + \ddot{y} \dot{x} = 2\dot{T}
\]

cartesian coordinates

\[
E = \sum \frac{1}{2} m_i \dot{q}_i^2 + U(q_1, ..., q_T)
\]

2F) Momentum

homogeneity of space: properties of closed systems are invariant under translations

\[
\xi \to \xi + \Delta \xi \Rightarrow \delta L = \sum \frac{\partial L}{\partial \dot{q}_a} \Delta \dot{q}_a = \sum \frac{\partial L}{\partial \dot{q}_a} \Delta \dot{q}_a
\]

\[
\delta L = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_a} = 0
\]

\[
\Rightarrow \frac{\partial L}{\partial \dot{q}_a} = 0
\]

generalized momentum \( p_a = \frac{\partial L}{\partial \dot{q}_a} \)

is conserved

if we have a system of particles

then \( \mathbf{P} \rightarrow \mathbf{P} + \mathbf{F} \) is an invariance
true $\Rightarrow \delta \phi = \frac{d}{dt} (\vec{p} \times \vec{l}) = 0$

$$\Rightarrow \vec{l} = \vec{r} \times \vec{p} \text{ is conserved}$$

For $n$ particles $\delta \phi \times \vec{p}_k = \delta \vec{r}_k$

then $\vec{l} = \sum_k \vec{r}_k \times \vec{p}_k$ is conserved

$$\vec{r}_k = \frac{\vec{r}_k}{\mu} + \vec{a} \Rightarrow \vec{l} = \vec{l}' + \vec{a} \times \vec{p}$$

$L_z \in$ cylindrical coordinates

$$L_z = xy - y \dot{x}$$

$$= m (xy - y \dot{x})$$

$$= m (r^2 \cos \phi \cos \theta \dot{\phi} + \sin^2 \theta \dot{\theta})$$

$$= mr^2 \dot{\phi}$$
Shape of a rope

Length of rope $L > 2a$
Mass density $\rho$
Assume shape function $f(x)$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx \sqrt{1 + f'(x)^2}$$

$$\int_{-a}^{a} ds = L \text{ constraint} = \int_{-a}^{a} dx \sqrt{1 + f'(x)^2}$$

The rope hangs such that the potential energy is minimized subject to this constraint.

Potential energy

$$V = \int_{-a}^{a} ds \cdot f(x) \mu g$$

$$= \int_{-a}^{a} dx \cdot f(x) \sqrt{1 + f'(x)^2} \mu g$$

Functional

$$F(f) = \int_{-a}^{a} dx \cdot \mu g \sqrt{1 + f'(x)^2} - \mu g \int_{-a}^{a} dx \sqrt{1 + f'(x)^2}$$

Lagrange eq.

$$E L \text{ eq.} \quad \frac{d}{dx} \left[ \frac{d}{df} \left( \frac{f'}{(1 + f'^2)^{3/2}} \right) \right] = -\mu g$$

$$\Rightarrow \frac{d}{dx} \int \left( \frac{f'}{(1 + f'^2)^{3/2}} \right) - \mu g \sqrt{1 + f'^2} = 0$$

$$-\frac{f''(\mu g f - \lambda)}{\sqrt{1 + f'^2}} - \frac{(\mu g f - \lambda) f'^2 f''}{(1 + f'^2)^{3/2}} - \mu g \sqrt{1 + f'^2} = 0$$
\[ f''(xg + f(x)) + f'(xg + f(x))f''(f'' + f''') = 0 \]

\[ f''(xg + f(x))(1 + f'' + f'''^2) - mg = 0 \]

\[ f''(xg + f(x)) = mg (1 + f'' + f''') \]

\[ m g (f'' + f''') - f'' = m g \]

**Trial solution:** \( f = A \cos \lambda x + B \cos \lambda x \)

\[ B m A \frac{\lambda}{\sin \lambda x} + m g (A^2 - \lambda^2) - \lambda A \sin \lambda x = m g \]

**Constraint:** \( \int_{a}^{\infty} ax \sqrt{1 + \sin^2 \lambda x} \, dx = L \)

\[ \Rightarrow m g \lambda^2 \lambda^2 = m g \Rightarrow A^2 \lambda^2 = 1 \quad (1) \]

\[ B m A \sin \lambda x = \lambda A \lambda \lambda \Rightarrow B = \frac{\lambda}{m g} \quad (2) \]

\[ \int_{-a}^{\infty} \sin \lambda x \, dx = \int_{-a}^{\infty} \cos \lambda x \, dx \]

\[ = \frac{1}{2} \sin \lambda x \bigg|_{-a}^{\infty} = L \]

\[ = \frac{1}{2} \sin \lambda x \bigg|_{-a}^{\infty} = L \quad (3) \]

\[ f(a) = f(-a) = A \cos \lambda a + B \quad (4) \]

4 eqs for 4 unknowns: \( A, B, \lambda, \lambda \)
(ii) The Brachistochrone

What is the shape of a path that minimizes the time to go from 1 to 2 starting from rest from 1 in a gravitational field with acceleration \( g \)?

\[
\frac{1}{2} m v^2 = (f(0) - f(x)) g
\]

\[
f(0) - f(x) > 0
\]

otherwise \( m \) never gets to \( x \) to 2

\[
v^2 = 2 (f(0) - f(x)) g
\]

the time it takes is

\[
T = \int \frac{dx}{v(x)} = \int \frac{dx}{x \sqrt{1 + f'(x)^2}}
\]

\[
= \int \frac{dx}{x \sqrt{1 + f'(x)^2}} \frac{Vf(0)}{Vf(x) - f(0)}
\]

minimizing \( T \) gives

\[
\frac{d}{dx} \left( \frac{\partial F}{\partial F'} - \frac{\partial F}{\partial F} \right) = 0
\]

\[
\frac{d}{dx} \frac{f}{\sqrt{1 + f'(x)^2}} - \frac{\sqrt{1 + f'(x)^2}(f')}{(f^2 - f(0)^2)^{3/2}} = 0
\]

\[
f_0 - f \equiv f
\]
\[
\frac{d}{dx} \left( \frac{f'}{\sqrt{1+f'^2}} \right) - \frac{\sqrt{1+f'^2}}{f'^2} = 0.
\]

Instead of looking for an explicit solution we first exploit that \( F = F(f, f') \)

then \( \frac{dF}{dx} = \frac{\partial F}{\partial f} f' + \frac{\partial F}{\partial f'} f'' \)

\[
= \frac{d}{dx} \left( \frac{\partial F}{\partial f'} f' \right) + \frac{\partial F}{\partial f'} f''
\]

\[
= \frac{d}{dx} \left( F - \frac{\partial F}{\partial f'} f' \right) = 0
\]

\[
\Rightarrow F - \frac{\partial F}{\partial f'} f' = \alpha
\]

\[
F = \frac{\sqrt{1+f'^2}}{\sqrt{f_0 - f}} \Rightarrow F - \frac{\sqrt{1+f'^2}}{\sqrt{f_0 - f}} = \frac{1}{\sqrt{f_0 - f}} f' = \alpha
\]

\[
\Rightarrow f'^2 = (1+\alpha^2) = \frac{1}{\alpha^2} f_0 - f
\]

\[
= \frac{1}{\sqrt{1+f'^2} \sqrt{f_0 - f}} = \alpha
\]

\[
f_0 - f \equiv \hat{f} \Rightarrow (1+\alpha^2) \hat{f} = \frac{1}{\alpha^2}
\]

\[
\frac{d}{dx} \left( \frac{1}{\alpha^2 f' - 1} \right)^{1/2} = \frac{1}{\alpha^2} \Rightarrow \left( \frac{1}{\alpha^2 f' - 1} \right)^{1/2} = \frac{1}{\alpha^2} \frac{1}{\alpha} \sin \left( \theta - \frac{\alpha}{\alpha^2} \right)
\]

\[
\frac{d}{dx} \left( \frac{1}{\alpha^2 f' - 1} \right)^{1/2} = dx \Rightarrow \frac{1}{\alpha} \sin \left( \theta - \frac{\alpha}{\alpha^2} \right)
\]

\[
\hat{f} = \frac{1}{\alpha} \left( 1 - \omega \hat{f} \right)
\]

... CYCLOID...
2.1) Cyclic coordinates

Coordinate $q_k$ is called a cyclic coordinate if the Lagrangian does not depend on $q_k$.

Then \[ \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} = 0 \]

\[ \Rightarrow \frac{\partial L}{\partial \dot{q}_k} \text{ is a constant of motion} \]

Example: central force in plane

\[ L = \frac{1}{2m} \left( r^2 + r^2 \dot{\theta}^2 \right) - V(r) \]

\[ \frac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} \text{ is conserved} \]

\[ \uparrow \]

Angular momentum in plane

2.2) Noether's theorem

This theorem relates symmetries and conserved quantities.

We consider a coordinate transformation such that $L$ does not depend on the parameter of this transformation.
\[
q_\lambda(x) = F_\lambda(q_1, \ldots, q_n, x)
\]

\[L_\lambda = L(q_\lambda(x), q_\lambda(x), \lambda) \text{ is independent of } \lambda\]

\[
\frac{d}{dx} L_\lambda = 0
\]

\[
\frac{d}{dx} (\frac{\partial}{\partial q_k(x)}) L_\lambda + \frac{\partial}{\partial q_k(x)} \frac{d}{dx} L_\lambda
\]

\[
= \left( \frac{d}{dt} \frac{\partial}{\partial q_k(x)} \right) \frac{dq_k(x)}{dt} + \frac{\partial}{\partial q_k(x)} \frac{d}{dt} \frac{dq_k(x)}{dt}
\]

\[
= \frac{d}{dt} \left( \frac{\partial}{\partial q_k(x)} \right) \frac{dq_k(x)}{dt} = 0
\]

\[
\Rightarrow \frac{\partial}{\partial q_k(x)} \frac{dq_k(x)}{dx} = \text{constant}
\]

**Noether's theorem:** \( \lim_{\lambda \to x} \frac{\partial}{\partial q_k(x)} \frac{dq_k(x)}{dx} \) in conserved

**Example:** \( L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \)

Rotation about z axis

\( x(\theta) = x \cos \theta + y \sin \theta \)

\( y(\theta) = -x \sin \theta + y \cos \theta \)

\[
\frac{dx(\theta)}{d\theta} = -x \sin \theta + y \cos \theta
\]

\[
\frac{dy(\theta)}{d\theta} = -x \cos \theta - y \sin \theta
\]

\[
L_\theta = \frac{1}{2} m (\dot{x}(\theta)^2 + \dot{y}(\theta)^2 + \dot{z}^2)
\]

\[
(\dot{x}(\theta), \dot{y}(\theta)) = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)
\]

\[
\Omega_{\theta} \left[ \frac{d}{d\theta} \frac{\partial L}{\partial \dot{q}_k} + \frac{\partial L}{\partial q_k} \right] = m \left[ \dot{x} y + \dot{y} x \right]
\]
3 a) Motion in 1d

Cartesian coordinates

\[ L = \frac{1}{2} m \dot{x}^2 - U(x) \]

\[ \Rightarrow \frac{dx}{dt} = \sqrt{2m(E-U)} \]

\[ E = \frac{1}{2} m \dot{x}^2 + U(x) \]

\[ t = \int \frac{dx}{\sqrt{2m(E-U)}} \]

Eqs. of motion in 1d can always be integrated

2 turning points =
motion is bounded for such type potential

3 b) Reduced mass

\[ \lambda = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(\mathbf{r}_1 - \mathbf{r}_2) \]

\[ R = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \]

\[ r = \mathbf{r}_1 - \mathbf{r}_2 \]

\[ r_1 = \frac{m_1 \mathbf{r}_1}{m_1 + m_2} \]

\[ r_2 = -\frac{m_2 \mathbf{r}_2}{m_1 + m_2} \]

\[ L = \frac{1}{2} \left( \frac{m_1 \dot{r}_1}{m_1 + m_2} \right)^2 + \frac{1}{2} \left( \frac{m_2 \dot{r}_2}{m_1 + m_2} \right)^2 - U(r) \]

\[ = \frac{1}{2} \lambda \dot{r}^2 - U(r) \]

Reduced mass

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \]
3.(c) Motion in a central field

\[ u = u(r, \frac{\theta}{r}, \frac{\phi}{r}) \Rightarrow \text{problem can be reduced to relative coordinate} \frac{\theta}{r} = \frac{\phi}{r} = 0 \]

Lagrangian in spherical coordinates

\[ L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta \right) - u(r, \theta, \phi) \]

\[ L \text{ is invariant for rotations about} \ r = 0 \]

\[ \theta \rightarrow \theta + \delta \theta, \ \phi \rightarrow \phi + \delta \phi, \ r \rightarrow r = \delta L = 0 \]

\[ \Rightarrow L = \vec{r} \times \vec{p} \text{ is conserved}, \ L_z = m r^2 \dot{\phi} \]

\[ \vec{F} = L = \text{motion is in a plane} \]

choose \ \theta = 50^\circ \text{ for this plane} \]

\[ \Rightarrow L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - u(r, \theta) \]

\[ \phi \text{ is cyclic coordinate} \Rightarrow \frac{\partial \phi}{\partial \phi} = 0 \]

\[ \dot{\phi} \delta \phi = m r^2 \dot{\phi} \delta \phi \text{ is conserved} \]

\[ \text{area} \ \delta A = r^2 \phi \delta \theta \]

\[ = \frac{\delta A}{\delta \phi} = r^2 \dot{\phi} = \text{constant} \]

(Corresponds to the second raw)
\( L = m r^2 \dot{\phi} \)

\[
E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + U(r)
\]

\[
= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \frac{L^2}{m r^2} + U(r)
\]

\[
= \frac{1}{2} m \frac{L^2}{m r^2} + \frac{L^2}{2m r^2} + U(r)
\]

\[
V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r)
\]

Centrifugal potential

1st motion in effective potential \( V_{\text{eff}}(r) \) for \( r \).

\[
V_{\text{eff}}(r)
\]

\[
\min \quad \max
\]

Radial motion

Particle moves in between the turning points.

\[
0 = r = \left( \frac{m}{E - V_{\text{eff}}} \right)^{\frac{1}{2}}
\]

\[
\Rightarrow \frac{dr}{dt} = \left( \frac{2m}{m (E - V_{\text{eff}})} \right)^{\frac{1}{2}}
\]
\[ t = \int_{r_{\min}}^{r_{\max}} \frac{dr}{\left(\frac{r^3}{m^2} (E-V_{eff})\right)^{\frac{1}{2}}} \]

For a closed orbit, the period is
(if center is at \( r = 0 \))
\[ T = 4 \int_{r_{\min}}^{r_{\max}} \frac{dr}{\left(\frac{r^3}{m^2} (E-V_{eff})\right)^{\frac{1}{2}}} \]

3e) Condition for a closed orbit

\[ \frac{d\phi}{dt} = \frac{L_z}{mr^2} \Rightarrow d\phi = \frac{L_z}{mr^2} \frac{dr}{\left(\frac{r^3}{m^2} (E-V_{eff})\right)^{\frac{1}{2}}} \]

\[ 0 < \phi = \int_{r_{\min}}^{r_{\max}} \frac{L_z \, dr}{m \, \left(\frac{r^3}{m^2} (E-V_{eff})\right)^{\frac{1}{2}}} \]

Orbit is closed if \( \phi = \frac{2\pi m}{n} \)
\( m, n \in \mathbb{Z} \)

\( \phi \) is a continuous function of \( E \), therefore all orbits can be closed only if \( \phi \) is independent of \( E \).

So \( \phi \) should be the same for \( E \to 0 \) and for \( E \to \infty \). We are going to analyze these limits.
we first consider the minimum value of $E$, i.e., an almost circular orbit.

Then we can expand the potential to 2nd order

$$V_{eff}(r) = V_{eff}(r_0) + \frac{1}{2} (r - r_0)^2 V''(r_0)$$

where we have introduced the variable $u = \frac{r}{r_0}$ so that

$$\phi = \int_{\text{unit Leda}} \frac{\text{unit Leda}}{\sqrt{2m(E - V_{eff}(r))}}$$
\[\phi = S \frac{du}{\sin \theta} \left( \frac{2m}{L^2} (E - V_{\text{eff}}(u)) - \frac{1}{2}(u - m)^2 - V_{\text{eff}}(u) \right)^{1/2}\]

motion of a particle in \( V_{\text{eff}}(u) \) with mass \( \frac{L^2}{m} \)

to 2nd order we have harmonic motion

with \( \omega = \frac{V_{\text{eff}}''(u_0)}{(L^2/m)} \)

\[\Rightarrow \phi \text{ is } \frac{1}{2} \text{ period}\]

\[\Rightarrow \phi = \frac{2\pi}{\omega} = \frac{2\pi}{L^2/m}\]

\[V_{\text{eff}}''(u) = \frac{L^2}{m} + u''(u_0)\]

\[V_{\text{eff}}(u_0) = 0 = \frac{L^2}{m} u_0 + u'(u_0) = \frac{L^2}{m} u_0 = -u'(u_0)\]

\[\Rightarrow V_{\text{eff}}''(u_0) = \frac{\frac{L^2}{m} + u''(u_0)}{\frac{L^2}{m}} = -u'' + u_0 u''\]

all orbits can only be closed if \( \phi \)
is independent of \( \frac{L^2}{m} \). This is the case

if \( \omega \) does not depend on \( u_0 \).

This is the case if \( u_0 \) is a homogeneous
function of \( u_0 = 1 \) \( u = 2u_0 \)

\[w = \frac{u + u''(u_0)}{u} = k + 2\]

\[\Rightarrow \phi = \frac{\pi}{2w} = \frac{\pi}{2(k+2)}\]

closed if \( \phi \in \Phi \Rightarrow k = \frac{1}{2} \)
Now we know that $V = 2 \frac{1}{r}$ and take

and we consider $E \rightarrow 0$.

Then $\text{Min} = \frac{d \mathcal{V}}{E \kappa}$ \hspace{1cm} \text{Max} = \frac{2 \kappa E}{L^2}$ \hspace{1cm} \Rightarrow \phi = \int \text{Max} \frac{L^2 da}{\sqrt{2h} \left( \frac{L^2}{2m} \text{Max} - \frac{L^2}{2m} \text{Min} - 2 \kappa \frac{L^2}{2m} \right)^{1/2}} \hspace{1cm} \text{Max} \rightarrow 0 \Rightarrow \phi = \int \text{Max} \frac{da}{\sqrt{\text{Max} - \text{Min}}} = \frac{\pi}{2}$

$\Rightarrow \kappa = 2$ : harmonic oscillator.

For $V < 0$ we need $a < 0$ in order to have bound states.

$E \rightarrow 0 \Rightarrow \text{Min} = 0$ \hspace{1cm} \frac{L^2}{2m} \text{Max} + 2 \kappa \frac{L^2}{2m} \text{Max} = 0

\Rightarrow \text{Max} = - \frac{2mL^2}{L^2} \hspace{1cm} a = \frac{a + 2}{2m} \sqrt{-2a} \sqrt{\frac{L^2}{2m} a}$

$\Rightarrow \phi = \sqrt{\frac{L^2}{2m} a} \text{Max} \sqrt{\frac{L^2}{2m} a} \sqrt{\frac{L^2}{2m} a} \Rightarrow \text{Max} \Rightarrow \kappa = 1$ : Kepler problem

$\text{other condition} \Rightarrow \phi = \frac{\pi}{\sqrt{a + 1}} \sqrt{\frac{L^2}{2m} a}$
24) What is special for the Kepler problem and the nO?

\[ V(r) = \frac{2}{r} \quad \text{are special} \]
\[ V(r) = -\frac{K}{r} \]

**Reason** there is an additional symmetry

the additional symmetry is in the direction of the long axis.
If the direction of the long axis is changing the orbit will be a rosette

For the Kepler problem the direction of the long axis is given by the Runge-Lenz vector
\[ \mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{Km^2}{r} \]
we will show in the homework that
\[ \frac{dA}{dt} = 0 \]

For an orbit that closes after 2\pi \( 6\pi \) is a single valued function of \( r \). This can happen if the equations of motion can be solved algebraically. There is no reason to expect a single valued function from integration.
Indeed, for the Kepler (or ho) problem we have three independent conserved quantities:

\[
\begin{align*}
L & = (r, \theta, \dot{r}, \dot{\theta}) = \mathcal{L} \\
E & = (r, \theta, \dot{r}, \dot{\theta}) = \mathcal{E} \\
\mathbf{A} & = (r, \theta, \dot{r}, \dot{\theta}) = \mathbf{a}
\end{align*}
\]

This implies that we can eliminate \( r \) and \( \theta \) resulting in an equation that relates \( \dot{r} \) and \( \dot{\theta} \) algebraically, i.e., \( F(r, \theta) = 0 \).

For simple enough problems we expect this to have a single valued solution.

\[ \text{(34) Kepler orbits} \]

\[
\begin{align*}
\dot{\theta} & = \sqrt{\frac{2m}{r} (E - \frac{L^2}{r^2})} \\
\frac{dr}{d\theta} & = \frac{L^2}{r^2} \\
\frac{dx}{\sqrt{a + bx + cx^2}} & = -\frac{1}{\sqrt{b}} \cos^{-1} \left( \frac{b + 2cx}{\sqrt{b^2 - 4ac}} \right) \\
\cos(\theta - \phi) & = \frac{4\dot{r}}{r} - \frac{m^2}{2L^2} \\
\text{can be put equal to zero}
\end{align*}
\]
If \( p = \frac{L e}{m^2} \), then \( \frac{p}{r} = 1 + e \cos \phi \) is an ellipse.

\[
\begin{align*}
e &= \sqrt{1 + \frac{e \xi}{m d^2}} \\
e &= \text{eccentricity} \\
a &= \frac{p}{1-e^2} \quad \text{major semi-axis} \\
b &= \frac{p}{\sqrt{1-e^2}} \quad \text{minor semi-axis}
\end{align*}
\]

\[
\begin{align*}
r &= \frac{p}{1+e \cos \phi} \\
r_{\text{max}} &= \frac{p}{1-e} \\
r_{\text{min}} &= \frac{p}{1+e}
\end{align*}
\]

\[
\begin{align*}
r_{\text{max}} - r_{\text{min}} &= \frac{2 e p}{1-e^2} \\
r_{\text{max}} + r_{\text{min}} &= \frac{2 p}{1-e^2}
\end{align*}
\]

\[
\begin{align*}
\text{period} : \quad \frac{\Delta \phi}{\Delta t} &= \frac{e}{2m} \\
\int dA &= \int_0^T \frac{L e}{2m} dt = \frac{L e}{2m} T \\
\pi ab &= \frac{p^2}{(1-e^2)^{3/2}} = \frac{p^2 a^{3/2}}{(1-e^2)^{3/2}}
\end{align*}
\]

\[
\begin{align*}
T &= \frac{2 \pi}{\sqrt{\frac{m^2}{2} a^{3/2}}} \\
\text{Kepler's 3}
\end{align*}
\]
$E > 0, \ e > 1$: Hyperbola

\[ \frac{p}{r} = 1 + e \cos \varphi \]

\[ \Rightarrow r = \frac{p}{1 + e \cos \varphi} \]

\[ r_{\text{min}} = \frac{p}{1 + e} \]

parabola for $e = 1$. 
Scattering

\[ \frac{d\sigma}{d\Omega} \]

\[ \Omega \] is the number of particles per unit time per unit area per beam.

\[ d\sigma = \text{number of particles scattered into } d\Omega \text{ per unit time} \]

Cross section

\[ \sigma(\theta, \phi) = \frac{1}{\Omega} \frac{d\sigma}{d\Omega} \]

Total cross-section

\[ \sigma_{\text{tot}} = \int d\Omega \sigma(\theta, \phi) \]

\[ d\Omega = \sin\theta \; d\theta \; d\phi \]

We now work out the case of axial symmetry.

Impact parameter

\[ |L| = mv - b \]

Number of particles scattered into \( \theta, \theta + d\theta \) per unit of incoming flux per unit time is

\[ \frac{d\sigma}{d\Omega} = 2\pi \sigma(\theta) \sin\theta \; d\theta \]
\[ \sigma(\theta) d\theta = \left[ \int_0^{2\pi} \sigma(\theta, \phi) \sin \theta \ d\phi \right] d\theta \]
\[ = 2\pi \sigma(\theta, \phi) \sin \theta \]

\( \uparrow \) does not depend on \( \phi \) in case of axial symmetry

\[ \Rightarrow \frac{d^2}{d\theta} = \int_0^{2\pi} \sigma(\theta, \phi) \ d\phi \]

We can also count the number of particles from the incoming side:

\[ -2\pi b \ dB = dS \Rightarrow \]

\( \uparrow \) minus sign because \( \theta \) decreases if \( b \) increases

\[ -2\pi b \ dB = 2\pi \sigma(\theta) \sin \theta \ d\theta \]

\[ \Rightarrow \sigma(\theta) = -\frac{b(\theta)}{\sin \theta} \frac{dB}{d\theta} \]

(b) Rutherford cross-section

We consider a central potential and calculate \( b(\theta) \).

First, we derive a general expression and then work out the result for a 1/r potential.
\[
\theta(r) = \pi - \sum \frac{\sqrt{\frac{L^2}{2m}}}{r^2 \sqrt{E - V(r) - \frac{L^2}{2m}}} \, dr
\]

For \( \theta(r) > 180^\circ \) (lower branch)

Scattering angle \( \theta \)

\[ + \theta = -\pi + \theta(\infty) = +\pi - 2 \sum \frac{\sqrt{\frac{L^2}{2m}}}{r^2 \sqrt{E - V(r) - \frac{L^2}{2m}}} \, dr \]

attractive scatterer \( \theta_+ (r \rightarrow \infty) < \frac{\pi}{2} \)

\( \theta < 0 \) attractive

\( \theta > 0 \) repulsive
We now calculate \( b(\theta) \) for \( V(r) = -\frac{k}{r} \)

\[
\theta = \pi - 2 \int_c^\infty \frac{dr}{r \sqrt{r^2 (1 - V(r)/E)^2}}
\]

\[
= \pi - 2 \int_0^{\infty} \frac{a_0 \, da}{\sqrt{a_0^2 (1 + \frac{a_0}{E})}}
\]

\[
= \pi - 2 \int_0^{\infty} \frac{a_0 \, da}{\sqrt{a_0 (1 - \frac{a_0}{E})}}
\]

\[
\frac{L_2}{2mE} = \frac{\hbar^2 \omega^2 \xi}{m^2 \omega^2} = b^2
\]

\[
E = \frac{1}{2} m \omega^2 r_0^2
\]

\[
L_2 = m \omega b
\]

\[
r_0^2 + \frac{2 \hbar \omega}{E} = 0
\]

\[
r_0 = -\frac{\hbar}{E} \sqrt{\frac{\hbar^2}{E^2} + \frac{\hbar^2}{E^2}}
\]

\[
\frac{V(r) + \frac{L_2}{2mE}}{r_0}
\]

\[
\int \frac{a_0 \, da}{\sqrt{a_0 (1 - \frac{a_0}{E})}}
\]

\[
\theta = \pi - 2 \cos^{-1} \left( \frac{a_0 - a_1}{a_0 - a_1} \right) + 2 \cos^{-1} \left( \frac{a_1 + a_4}{a_0 - a_1} \right)
\]

\[
= \pi - 2 \pi + 2 \cos^{-1} \left( \frac{-\sqrt{2}E}{\frac{\hbar^2}{E^2} + \frac{\hbar^2}{E^2}} \right)
\]

\[
\Rightarrow \cos \left( \frac{\pi}{2} + \frac{\theta}{2} \right) = -\frac{\sqrt{2}E}{\sqrt{\frac{\hbar^2}{E^2} + \frac{\hbar^2}{E^2}}}
\]

\[
\Rightarrow \sin \frac{\theta}{2} \left( \frac{\hbar^2}{E^2} + \frac{\hbar^2}{E^2} \right) = \frac{\sqrt{2}E}{E^2}
\]
\[ 2 \left( \frac{1}{b} \right)^2 = \frac{x^2}{E^2} \frac{\cos^2 \frac{x}{2}}{\sin^2 \frac{x}{2}} \implies b = \frac{x}{2E} \cot \frac{\theta}{2} \]

Cross-section

\[
\delta(\theta) = -\frac{b(\theta)}{\sin \theta} \frac{dx}{d\theta}
\]

\[
= -\left( \frac{a^2}{2E} \right) \frac{\cot \frac{\theta}{2}}{\sin \theta} \frac{1}{2} \left( -1 - \frac{\cos^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \right)
\]

\[
= \frac{a^2}{16E^2} \frac{1}{\sin^2 \frac{\theta}{2}}
\]

Rutherford cross section
4 a) \textbf{Rotations}

Rotation is a transformation that leaves the scalar product invariant:

\[ x \cdot y = \sum_{i=1}^{n} x_i \cdot y_i \]

\[ x_i \rightarrow D_{ik} x_k \]

\[ y_i \rightarrow D_{ik} y_k \]

Invariant if \( \sum_{i=1}^{n} D_{ik} D_{ie} = \delta_{ke} \)

or \( D^T D = 1 \)

\( D^T D = 1 \Rightarrow \det D^T \det D = 1 \)

\( \det D = \pm 1 \)

\( \det D = -1 \) : Reflections

4 b) \textbf{Rotations in the plane}

We can either rotate a vector by \( \theta \) or the coordinate system by \(-\theta\).

Generally, we will think of transforming the coordinate system.
\[ y' = x' \cos \theta - y' \sin \theta \]
\[ x' = x' \sin \theta + y' \cos \theta \]

\[ R_z(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \]

Rotation about \( y \)-axis

\[ R_y(\theta) = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{pmatrix} \]

Rotation about \( x \)-axis

\[ R_x(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix} \]
Infinite:esimal transformation

\[ R_3(\varepsilon) = 1 + \varepsilon \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \varepsilon M_3 \]

\[ R_2(\varepsilon) = 1 + \varepsilon \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \varepsilon M_2 \]

\[ R_1(\varepsilon) = 1 + \varepsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \varepsilon M_1 \]

\[ M_1 M_2 - M_2 M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = M_3 \]

\[ \text{same} \quad [M_2, M_3] = M_1 \]
\[ [M_3, M_1] = M_1 \]

Same commutation relation as the angular momentum operator

\[ R_1(\theta) = \left(1 + \frac{\theta M_1}{n}\right)^n = e^{\theta M_1} \]

General rotation \[ R(\Theta) = e^{\sum_{\alpha=1}^{3} \Theta_\alpha M_\alpha} \]
Let \( u = \sqrt{x^2 + y^2} \)

\[
\begin{align*}
x &= u \cos \theta \\
y &= u \sin \theta \\
x' &= a \cos(x+y) \\
y' &= a \sin(x+y)
\end{align*}
\]

\[
\begin{align*}
x' &= u \left[ \cos^2 \theta \cos y - \sin \theta \sin \theta \sin y \right] \\
y' &= u \left[ \sin \theta \cos \theta \cos y + \sin \theta \sin \theta \cos y \right]
\end{align*}
\]

\[
\begin{align*}
x' &= x \cos \phi - y \sin \phi \\
y' &= y \cos \phi + x \sin \phi
\end{align*}
\]

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

O(2) rotations are matrices of the form \( \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \)

This set of matrices is a Lie group (it is continuous). Then a rotation by \( \theta \) can be viewed as \( n \) successive rotations by \( \theta/n \).

\[
O(\theta) = \left[ O \left( \frac{\theta}{n} \right) \right]^n
\]

\[
\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = 1 + \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

infinitesimal generator

Lie groups are determined by the properties of the infinitesimal generator.
5a Rigid body motion

\text{x,y,z : inertial frame}
\text{x_1,x_2,x_3: body fixed frame}
\text{origin in cm}

Rigid body has 6 dof!
\begin{align*}
\dot{\mathbf{r}} &= \dot{\mathbf{R}} + \dot{\phi} \times \mathbf{R} \\
\end{align*}

\text{motion of point P :}
\begin{align*}
\dot{\mathbf{r}_p} &= \dot{\mathbf{R}} + \dot{\phi} \times \mathbf{r}_p \\
\end{align*}

\text{\dot{\phi} = } \begin{align*}
\frac{d\phi}{dt} &= \frac{d\mathbf{R}}{dt} + \frac{d\phi}{dt} \times \mathbf{R} \\
\mathbf{V}_p &= \mathbf{V} + \mathbf{v}_p \\
\mathbf{v}_p &= \frac{d\mathbf{r}_p}{dt} \\
\end{align*}

\text{cm velocity}

\text{This decomposition is also valid if O is not the cm, but the velocity of O is of course different.}

\begin{align*}
\mathbf{r} &= \mathbf{r}_1 \times \frac{\dot{\mathbf{a}}}{t} \\
\mathbf{V}_p &= \mathbf{V} + \mathbf{v}_p \\
\Rightarrow \mathbf{V}_1 &= \mathbf{V} + \mathbf{v}_p \\
\Rightarrow \mathbf{V}_1 &= \mathbf{V} + \mathbf{v}_p \\
\Rightarrow \mathbf{v}_1 &= \mathbf{v}_p \\
\end{align*}

\text{angular or velocity of O}
If \( V + \omega \) then \( \frac{\partial V}{\partial t} = \omega \)\

\( V' \) is an axis through \( O' \), such axis is an instantaneous axis of rotation.

\[ T = \frac{1}{2} \sum m_r \left( V + \frac{\partial \mathbf{r}}{\partial t} \times \mathbf{r}_p \right)^2 \]

\[ T = \frac{1}{2} \sum m_r V^2 + \frac{1}{2} \sum m_r \left( \frac{\partial \mathbf{r}}{\partial t} \times \mathbf{r}_p \right)^2 \]

\[ \sum m_r \left( \frac{\partial \mathbf{r}}{\partial t} \times \mathbf{r}_p \right) = \mathbf{V} \times \mathbf{r}_p \cdot \sum m_r \frac{\partial \mathbf{r}_p}{\partial t} = 0 \]

\[ T = \frac{1}{2} M V^2 + \int d\mathbf{r} \mu(\mathbf{r}) \left( \frac{\partial \mathbf{r}}{\partial t} \times \mathbf{r} \right)^2 \]
\[
\sum_{ijk} \delta_{ij} \, r_j \, r_k \, \delta_{pr} \, r_p = r^2 - (r \cdot r)
\]

\[
\Rightarrow \quad \mathbf{I}_{\text{rot}} = \frac{1}{2} \sum_{pq} \left( \int d^3r \, \rho(r) \right) (\delta_{pq} \, x^2 - x_p x_q)
\]

\[
= \mathbf{I}_{\text{p}} = \frac{1}{2} \sum_{pq} \mu \, x_p x_q
\]

\[
= \frac{1}{2} \mathbf{I} \cdot \mathbf{V}^2 = \frac{1}{2} \sum_{pq} \mu \, x_p x_q \, \mathbf{I}_{\text{p}}
\]

\[
\mathbf{I}_{\text{p}} = \mathbf{I}_{\text{ex}} \Rightarrow \text{ I can be diagonalized}
\]

\[
\mathbf{I}_{\text{ex}} \cdot \mathbf{R} \cdot \mathbf{R}^T > 0 \quad \text{because } \mathbf{T} > 0
\]

If I is diagonal, then the body fixed frame is called a principle axis frame.

Then
\[
\mathbf{T} = \frac{1}{2} \mathbf{I}_1 \, \omega_1^2 + \frac{1}{2} \mathbf{I}_2 \, \omega_2^2 + \frac{1}{2} \mathbf{I}_3 \, \omega_3^2
\]

principle moments

Transformation of I under translation \( \mathbf{T} = \mathbf{T} + \dot{\mathbf{a}} \)

\[
\delta \mathbf{p} \cdot x^2 - \mathbf{p} \cdot x \mathbf{p} \rightarrow \delta \mathbf{p} \cdot x^2 - \mathbf{p} \cdot x \mathbf{p} + 2 \mathbf{r} \cdot \delta \mathbf{p} \sigma_{\mathbf{p}} + \delta \mathbf{p} \cdot \sigma_{\mathbf{p}} \mathbf{r}
\]

\[= \mathbf{a} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{p} \mathbf{a} - \mathbf{a} \cdot \mathbf{p} \mathbf{x}
\]

\[
5 \mathbf{r} \cdot x \, d^3r = 0 \quad \Rightarrow
\]

\[
\mathbf{I}_{\text{ex}} \rightarrow \mathbf{I}_{\text{ex}} + \delta \mathbf{p} \cdot \mathbf{a}^2 - \mathbf{a} \cdot \mathbf{p} \mathbf{x}
\]
Example

\[ \mu(r) = \frac{m \text{mass}}{2\pi r} \]

\[ I_{\mu\mu} = \int_0^L \mu(x) \left( \int_0^L r^2 - r_0 r \, dr \right) \, dx \]

\[ = \mu \frac{1}{3} L^3 \left( \int_0^L r^2 - r_0 r \, dr \right) \]

\[ = \mu \frac{1}{3} L^3 \left( \int_0^1 r^2 \, dr - \int_0^1 r \, dr \right) \]

\[ = \mu \frac{1}{3} L^3 \left( \frac{1}{3} - \frac{1}{2} \right) \]

\[ = \mu \frac{1}{3} L^3 \left( \frac{1}{6} - \frac{1}{2} \right) \]

\[ = \mu \frac{1}{3} L^3 \left( \frac{1}{6} \right) \]

\[ = \mu \frac{1}{3} L^3 \frac{1}{6} \]

\[ = \mu \frac{1}{3} L^3 \frac{1}{6} \]

\[ I = L^3 \begin{pmatrix} \frac{1}{6} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \]

Eigenvalues: \( \lambda_1 = 0 \), \( \lambda_2 = \frac{1}{3} \), \( \lambda_3 = \frac{1}{3} \)

\[ \lambda_1 = 0 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ in eigenvector} \]

\[ \lambda_2 = \frac{1}{3} \quad \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} \text{ in eigenvector} \]

or \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) in eigenvector \( x \)-axis

\[ \lambda = 0 \]

\[ \lambda = \frac{1}{3} \]

\[ \lambda = \frac{1}{3} \]
Any linear combination of $\left(\frac{1}{8}\right)$ and $\left(\frac{3}{8}\right)$ is also an eigenvector because of the degeneracy.

5c) Angular momentum

$$\vec{J} = \sum \text{d}r \times \mathbf{p} = \sum \text{d}r \times \left( \mathbf{\hat{r}} \times \mathbf{v} \right) + \sum \text{d}r \times (\mathbf{\hat{r}} \times \mathbf{\omega})$$

$$\sum \text{d}r \times \mathbf{v} = 0 \Rightarrow \text{an velocity does not contribute}$$

$$\mathbf{\hat{r}} \times (\mathbf{\hat{r}} \times \mathbf{\omega}) = \mathbf{\omega} \times \mathbf{\hat{r}}$$

$$= \mathbf{\omega} \left( \sum \text{d}r (\mathbf{\hat{r}} \times \mathbf{\omega}) \right)$$

$$= \mathbf{\omega} \left( \sum \mathbf{r} \times \mathbf{\omega} \right) = \mathbf{\omega} \left( \sum \mathbf{r} \mathbf{\omega} + \mathbf{\omega} \sum \mathbf{r} \right)$$

$$\Rightarrow \sum \mathbf{r} \mathbf{\omega} + \mathbf{\omega} \sum \mathbf{r} = \mathbf{I} \mathbf{\omega}$$

$$\Rightarrow \sum \mathbf{r} \mathbf{\omega} = \mathbf{I} \mathbf{\omega}$$

principal axis

$$\mathbf{I}_\perp = \mathbf{I}_x \mathbf{\omega}$$
1) Free spherical top: \( I_1 = I_2 = I_3 = I \)
   
   No forces \( \Rightarrow \\vec{M} = I \vec{\Omega} \)
   
   \( \vec{\Omega} \) constant \( \Rightarrow \vec{r}_2 \) constant

2) \( I_1 = I_2 > I_3 = 0 \)

   \[ I_{1m} = \delta_1 m \int d^3 \vec{x} \; x^2 \mu(x) = \delta_1 m \int d \vec{x}_1 \; x^2 \; \mu(x_1) \]

   \[ I_{2m} = \delta_2 m \int d^3 \vec{x} \; x^2 \mu(x) = \delta_2 m \int d \vec{x}_1 \; x^2 \; \mu(x_1) \]

   \( x_1, x_2 \) and \( x_2, x_3 \) terms vanish because \( x_1 = x_2 = 0 \)

   \[ \vec{J}_3 = I_3 \vec{\Omega}_3 = \delta_3 I_1 \vec{\Omega}_1 + I_2 \vec{\Omega}_2 \delta_3 \]

   \( \Rightarrow \vec{J}_3 = 0 \), \( \vec{\Omega}_3 = 0 \) rotation about axis \( \perp \vec{x}_3 \)

3) Symmetrical top free, no forces or torques
   
   \( \vec{M} = I_3 \vec{\Omega}_3 \) coordinate choice, \( x_3 \) in principle axis

   \( x_2 + M \)

   \( x_2 + x_3 \)

   \( \Rightarrow M_2 = 0 = I_2 \vec{\Omega}_2 \)

   \( \Rightarrow \vec{\Omega}_2 = 0 \)
\( M, \vec{z}, \text{ and } x_3 \) are co-planar

\[ \vec{v} = \vec{z} \times \frac{r}{I} = \text{axis of this plane} \]

but \( \vec{r} \) is fixed \( \Rightarrow \)

\( \Rightarrow \) axis of top rotates uniformly about \( \vec{M} \) in a circle. This motion is called precession.

\[ \Omega_1 = \frac{M}{I_3} \]

\[ \Omega_1 = \frac{M \cos \theta}{I_3}, \quad \Omega_1 = \frac{M}{I} = \frac{M \sin \theta}{I} \]

Next we calculate the precession velocity

\[ \Omega_{pr} = \vec{z} \times \frac{r}{I} = \Omega_{pr} \]

\[ \Omega_1 = \vec{v}_{pr} \parallel x_1 \]

\[ |\vec{v}_{pr}| = \Omega_1 r \]

\[ \vec{v}_{pr} = \frac{\vec{r}}{r \sin \theta} = \frac{M}{I} \]

**Interpretation**

The component of \( \vec{z} \) along \( \vec{x} \) does not contribute to precession.

\[ \Rightarrow \vec{v}_{pr} \parallel \vec{x}_1 \parallel \vec{M} \]

\[ \vec{v}_{pr} = \frac{\vec{r}}{r \sin \theta} = \frac{M}{I} \]
Eqs. of motion of a rigid body

Lagrangian \[ L = \frac{1}{2} m \dot{V}^2 + \frac{1}{2} I \omega \times \omega - U \]

cm coordinates \[ \dot{x}_1, \dot{x}_2, \dot{x}_3 \]
orientation \[ \phi_1, \phi_2, \phi_3 \]
\[ \omega x = \dot{\phi} \]

EL eqs. \[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \]
\[ \dot{x}_i = -\frac{\partial U}{\partial x_i} = F_i \]

Rotational eqs. \[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_i} - \frac{\partial L}{\partial \phi_i} = 0 \]
\[ \frac{d}{dt} I_i \dot{\phi}_i = -\frac{\partial U}{\partial \phi_i} \]
\[ \dot{\phi}_i \]
\[ \frac{d}{dt} \dot{\phi}_i = -\frac{\partial \Phi}{\partial \phi_i} \]

Change in potential energy by infinitesimal rotation:
\[ \delta U = \sum \sum \tau_k \delta \phi_k = -\sum \frac{\partial U}{\partial \phi_k} \delta \phi_k \]

Torque \[ \tau_k = \sum \sum \frac{\partial U}{\partial \phi_k} \dot{\phi}_k \times \dot{\phi}_k \]
\[ \tau_k = -\frac{\partial U}{\partial \phi_k} \dot{\phi}_k \]
\[ \frac{d}{dt} \tau_k = \kappa_k \]
\[ \kappa_k \]
\( \mathbf{r} = \mathbf{r}_1 + \mathbf{a} \) then \( \mathbf{\tau} = \mathbf{\tau}_1 + \mathbf{a} \times \mathbf{F} \)

\( \Rightarrow \frac{d\mathbf{\tau}}{dt} = \mathbf{0} \) then torque is independent of the choice of origin.

\( \Rightarrow \mathbf{\tau} \perp \mathbf{F} \) \( \exists \mathbf{a} \) such that \( \mathbf{\tau}' = 0 \)

\( \Rightarrow \mathbf{\tau} = \mathbf{a} \times \mathbf{F} \)
5) Euler angles (see Chapter 7 of Goldstein)

- $x_3$ vertical frame
- $x_2$ body fixed frame
- $x_1$ line of nodes: intersection of $x_1$ plane and $x_2$ plane.

Start with $x_1, x_2, x_3$ coinciding with $xyz$

i) rotate $x_1$ axis by $\phi$ about $z$

ii) rotate $x_3$ axis by $\theta$ about new $x_1'$

iii) rotate $x_1'$ by $\psi$ about $x_3$

We now express $z$ in $x_1'x_2'x_3'$ frame in the Euler angles.

We first express $\phi, \theta, \psi$ into $x_1'x_2'x_3'$ components.

$\dot{\psi} = (0, 0, \dot{\psi})$
$\dot{\theta} = (\dot{\theta} \cos \psi, -\dot{\theta} \sin \psi, 0)$

along line of nodes: moves in negative direction.

$\dot{\phi} = (\phi \sin \theta \sin \psi, \phi \sin \theta \cos \psi, \phi \cos \theta)$

along $z$ axis: projected on $x_1'x_2'$ plane.

Before rotating by $\Psi = \Pi \parallel x_2$.
\[ \begin{align*}
\theta &= \dot{\theta} \cos \gamma + \phi \sin \theta \sin \gamma \\
\dot{\gamma} &= -\dot{\phi} \sin \gamma + \phi \sin \theta \cos \gamma \\
\dot{\phi} &= \dot{\phi} + \phi \cos \theta \\
\text{(Symmetric case)} \\
\text{Example 1)} \quad \text{Free symmetric case} \quad I_1 = I_2
\end{align*} \]

\[ T_{\text{net}} = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\gamma}^2) + \frac{1}{2} I_2 \dot{\phi}^2 = \frac{1}{2} I_1 \left( \dot{\theta}^2 \sin^2 \theta + \dot{\gamma}^2 \right) + \frac{1}{2} I_2 \left( \dot{\phi}^2 + \dot{\theta} \cos \theta \right) \]

- \( x_1, x_2 \) axis in arbitrary \( \implies \) we can choose \( \gamma = 0 \)
- \( x_3 \) axis chosen along principal axis

\[ \text{no torque } \implies \frac{dM}{dt} = 0 \]

\[ M = (I_1 \dot{\theta}, I_2 \dot{\gamma}, I_3 \dot{\phi}) \]

\[ M_1 = \dot{I}_1 \dot{\theta} ; \quad \text{in body frame} \]

\[ M_2 = I_2 \dot{\gamma} = I_1 \dot{\theta} \sin \theta \]

\[ M_3 = I_3 \dot{\phi} = I_3 (\dot{\phi} \cos \theta + \dot{\gamma}) \]

Choose \( z \)-axis // \( \dot{M} \) \( \implies \frac{d\dot{M}}{dt} = 0 \)

\[ \psi = \text{constant} \implies \dot{M} \in x_2 \times \dot{x}_3 \] plane

\[ \begin{cases} 
\dot{M}_1 = \dot{M} \cos \psi \\
\dot{M}_2 = \dot{M} \sin \theta
\end{cases} \]

\[ \dot{\phi} = \frac{\dot{M}}{I_3} \quad \text{precession} \]

\[ I_3 \dot{\phi} = \dot{M} \cos \theta \]
Example 2)

moving and fixed
system have common origin

I_1, I_2, I_3, moments of
inertia wrt cm
\( \ddot{a} = (0, 0, 2) \)

\( x_3 \) axis is principal axis, choose \( \psi = 0 \)
(does not mean that \( \dot{\psi} = 0 \))

next moment of inertia wrt \( O \)

\[ I_1 \rightarrow I_1 + \alpha \ell^2 \]

\[ I_2 \rightarrow I_2 + \alpha \ell^2 \quad \text{choose } \psi = 0 \quad \text{(axis symmetry)} \]

\[ \Rightarrow L = \frac{1}{2} (I_1 + \alpha \ell^2) (\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \theta) \]

\[ + I_2 (\dot{\phi}^2 + \dot{\phi} \cos \theta)^2 - Mg \ell \cos \theta \]

\( \psi \) and \( \phi \) are cyclic \( \Rightarrow P_\psi, P_\phi \) are conserved

\[ P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_2 (\dot{\psi}^2 + \dot{\phi} \cos \theta) = I_2 \dot{\psi}^2 = M_3 \]

\[ P_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{(I_1 + \alpha \ell^2) \sin^2 \theta + I_3 (\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta}{M_3 \sin \theta} \quad \frac{I_3 \dot{\phi} \cos \theta}{M_3 \cos \theta} = M \dot{\phi} \quad \text{conserved.} \]
Energy is conserved

\[ E = T + U = \frac{1}{2} \left( I_1 + I_2 \right) \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\psi}^2 + \frac{1}{2} I_2 \dot{\phi}^2 + mg \ell \cos \theta \]

Eliminating \( \dot{\psi} \) and \( \dot{\phi} \) from \( E \)
give \( \dot{E} (\theta, \dot{\theta}) = \frac{1}{2} (I_1 + I_2) \dot{\theta}^2 + U_{gy}(\theta) \)

Result

Motion of top of top (nutation)
Euler Equations

\( \dot{\mathbf{\hat{r}}} \) in body fixed frame \( \frac{d\mathbf{\hat{r}}}{dt} = -\mathbf{\hat{r}} \times \mathbf{\hat{\omega}} \)

This is true for all body fixed vectors
\( \frac{d\mathbf{\hat{A}}}{dt} = -\mathbf{\hat{A}} \times \mathbf{\hat{\omega}} \)

When \( \mathbf{\hat{A}} \) changes in body fixed frame then
\[ \frac{d\mathbf{\hat{A}}}{dt} \bigg|_{bf} = \frac{d\mathbf{\hat{A}}}{dt} \bigg|_{bf} + \mathbf{\hat{\omega}} \times \mathbf{\hat{A}} \]

\[ \Rightarrow \quad \frac{d\mathbf{\hat{p}}}{dt} = \frac{d\mathbf{\hat{p}}}{dt} \bigg|_{bf} + \mathbf{\hat{\omega}} \times \mathbf{\hat{p}} \]

\[ \frac{d\mathbf{\hat{M}}}{dt} \bigg|_{bf} = \frac{d\mathbf{\hat{M}}}{dt} \bigg|_{bf} + \mathbf{\hat{\omega}} \times \mathbf{\hat{M}} \]

\( \mathbf{\hat{\omega}} \) (Torque)

Use principle axis in body fixed frame
\( \Rightarrow M_{\text{bf}} = I_{\text{bf}} \mathbf{\omega}_{\text{bf}} \) (no summation convention)

Eqs. of motion in body fixed frame

\[ \mathbf{F}_1 = \frac{d\mathbf{\hat{r}}}{dt} + \mathbf{\hat{\omega}} \times \mathbf{\hat{p}}_1 - \mathbf{\hat{\omega}} \times \mathbf{\hat{p}}_3 \]

\[ \mathbf{F}_2 = \frac{d\mathbf{\hat{r}}}{dt} + \mathbf{\hat{\omega}} \times \mathbf{\hat{p}}_2 - \mathbf{\hat{\omega}} \times \mathbf{\hat{p}}_3 \]

\[ \mathbf{F}_3 = \frac{d\mathbf{\hat{r}}}{dt} + \mathbf{\hat{\omega}} \times \mathbf{\hat{p}}_2 - \mathbf{\hat{\omega}} \times \mathbf{\hat{p}}_1 \]
\[ \begin{align*}
\dot{\kappa}_1 &= I_1 \frac{d\kappa_1}{dt} + (I_2 - I_3) \omega_2 \kappa_3 \\
\dot{\kappa}_2 &= I_2 \frac{d\kappa_2}{dt} + (I_3 - I_1) \omega_1 \kappa_3 \\
\dot{\kappa}_3 &= I_3 \frac{d\kappa_3}{dt} + (I_1 - I_2) \omega_1 \kappa_2 \end{align*} \]

Euler's equations

Example: Free symmetric top

\[ \kappa = 0 \]

\[ I_1 = I_3 \]

\[ \frac{d\kappa_2}{dt} = 0 \Rightarrow \kappa_2 = \text{constant} \]

\[ \frac{d\kappa_1}{dt} = -(I_2 - I_3) \omega_2 \kappa_3 = -\omega \kappa_2 \]

\[ \frac{d\kappa_3}{dt} = -(I_1 - I_2) \omega_1 \kappa_3 = -\omega \kappa_1 \]

\[ \Rightarrow \kappa_1 = A \cos \omega t, \quad \kappa_2 = A \sin \omega t \]

\[ M_y = I_1 \kappa_2 \]

In body fixed frame \( \Pi_3 = \text{constant} \)

but \( \Pi \) rotates about \( \kappa_3 \) with frequency \( \omega \)

In lab frame \( \Pi \) is fixed so in lab \( \kappa_3 \) rotates about \( \Pi \) with frequency \( \omega \)
\( \omega \) is the speed of the \( 1 \) and 2 axis

\[ |\omega| = |\dot{\phi}| \]

\[ \dot{\psi} = \frac{\Omega_3}{I_3} - \dot{\theta} \cos \theta \]

\[ = \frac{\Omega_3}{I_3} - \frac{M}{I_1} \cos \theta \]

\[ = \Omega_3 - \frac{I_3 R_3}{I_1} \]

\[ \rho_1 = \Omega_2 \]

\[ \text{Speed of } \rho_1 \text{-axis} \]

\[ \rho_1 \cos \theta + R_3 = R_3 - \frac{I_3 R_3}{I_1} \]

\[ \Rightarrow \rho_1 = - \frac{M}{I_1} \]

\( \rho_2 \)

\( \rho_3 \)

**Free asymmetric top**

\( I_3 > I_2 > I_1 \)

Free \( \Rightarrow \) \( E \) and \( \frac{1}{2} \) one conserved

\[ I_1 \rho_1^2 + I_2 \rho_2^2 + I_3 \rho_3^2 = 2E \]

\[ I_1 \rho_1^2 + I_2 \rho_2^2 + I_3 \rho_3^2 = M^2 \]

\( 6 \) dof

\( \gamma \) conserved quantities

\( \gamma \) can be integrated

We do not solve the eqns. of motion but

Study the motion in \( \gamma \) space
\( \Pi_1^2 + \Pi_2^2 + M_3 = M^2 \) Sphere
\[ \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{M_3}{I_3} = 2E \] Ellipsoid

Endpoint of \( \vec{r} \) moves on the intersection of the two

a) \( 2EI_1 < \Pi_2^2 < 2EI_3 \)

\( \Rightarrow \) Ellipsoid intersects the sphere
\( \Pi_2 \to 0 \), \( \Pi_3 \to 0 \)
\[ \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{M_3^2}{I_3} = 2EI_1, \]
\( \Pi_1^2 + \Pi_2^2 + M_3^2 = M^2 \)

\( \Rightarrow \)
\[ \Pi_2^2 \left(1 - \frac{I_1}{I_2}\right) + M_3^2 \left(1 - \frac{I_1}{I_3}\right) = \frac{3}{2}EI_1 > 0 \]

\( \sqrt{0} \)

This is the equation for an ellipse

\( \Rightarrow \) Motion is stable about the 1-axis.

b) \( \Pi_2^2 \to 2EI_3 \) then
\[ \frac{I_3}{I_1} \Pi_1^2 + \frac{I_2}{I_1} \Pi_2 + M_3 = 2EI_3 \]
\[ \Pi_1^2 + \Pi_2^2 + M_3^2 = M_2 \]

\( \Rightarrow \)
\[ \left(\frac{I_3}{I_1} - 1\right) \Pi_1^2 + \left(\frac{I_2}{I_1} - 1\right) \Pi_2^2 = 2EI_3 - M > 0 \]

\( \sqrt{0} \)

Ellipse, stable motion about 3 axis
c) \( \bar{E}^2 \rightarrow 2E \rightarrow 2 \rightarrow \bar{M}_1 \rightarrow 0, \bar{M}_2 \rightarrow 0 \)

\[
\left( \frac{I_{21}}{J_{11}} - 1 \right) \bar{M}^2_1 + \left( \frac{I_{22}}{J_{22}} - 1 \right) \bar{M}^2_2 = 2E \bar{I}_2 - \bar{M}^2
\]

Hyperbola about \( x_2 \)-axis
\( \Rightarrow \) unstable motion

\[ \bar{\nu}^2 = \bar{\nu}^2 + \bar{\nu}(t) \]

\[ \bar{\nu} = \bar{\nu} + \bar{\nu} \times \bar{r} \]

\[ \bar{m}_0 : L = \frac{1}{2} \bar{m} \bar{\nu}^2 - \bar{U}(\bar{r}) \]

\[ \bar{m}_1 : L = \frac{1}{2} \bar{m} (\bar{\nu}^2 + \bar{\nu}_1^2)^2 - \bar{U}(\bar{r}) \]

\[ = \frac{1}{2} \bar{m} \bar{\nu}^4 + \frac{1}{2} \bar{m} \bar{\nu}_1^4 + \bar{m} \bar{\nu} \bar{\nu}_1 \bar{\nu}^2 - \bar{U}(\bar{r}) \]

\[ EL \]

\[ \frac{d}{dt} \bar{m} (\bar{\nu}^2 + \bar{\nu}_1 \bar{\nu}) + \frac{\partial \bar{U}}{\partial \bar{\nu}_1} = 0 \]

\[ \Rightarrow \]

\[ \frac{d}{dt} \bar{m} \bar{\nu}_1 = -\frac{\partial \bar{U}}{\partial \bar{\nu}_1} - \bar{m} \frac{\partial \bar{\nu}}{\partial \bar{t}} \]

For a due to acceleration
In a frame \( \mathcal{U} \) the Lagrangian is

\[
L = \frac{1}{2} m (\dot{\mathbf{u}} + \mathbf{a})^2 + m u^2 \left( V - V_c(r) \right) + \frac{1}{2} m \ddot{r} \dot{V}^2 + \frac{1}{2} m \dot{V}^2 \]

which does not contribute to \( \delta L \).

\[
m \ddot{V} = \frac{d}{dt} \left( m \dot{r} \dot{V} \right) - m \ddot{r} \dot{V} + m V \frac{d V}{dr}
\]

Total derivative and does not contribute to \( \delta L \).

\[
\Rightarrow L = \frac{1}{2} m \dot{u}^2 + m \ddot{r} \dot{V}^2 + \frac{1}{2} m (2 \dot{r} \dot{V})^2 - m \ddot{r} \dot{V} + m V \frac{d V}{dr} - u(r)
\]

\[
m \ddot{r} \dot{V} = m \ddot{r} \dot{V} \quad \text{now in frame } \mathcal{U}
\]

\[
\frac{\partial L}{\partial \dot{u}} = m \dot{u} - m (\dot{r} \dot{V})
\]

\[
\frac{\partial L}{\partial \dot{r}} = m \ddot{r} \dot{V} + m \left( \begin{array}{c} \frac{d}{dt} \left( \mathbf{a} \right) \cdot \dot{V} \right)
\]

\[
f \frac{\partial L}{\partial \ddot{r}} = m \ddot{r} \dot{V} + m \left( \ddot{r} \dot{V} \right) + m \left[ \frac{d}{dt} \left( \mathbf{a} \right) \right] \dot{V} + m \left( \dot{r} \dot{V} \right) + m \left[ \ddot{r} \dot{V} \right]
\]

Force due to Coriolis force.

\[
\text{Force due to non-uniform rotation } - m \dot{u} \dot{u} - m \ddot{r} \dot{V}
\]
Small oscillations

Small oscillations about the equilibrium point are important in many applications. E.g. this gives the resonance frequency of a bridge.

\[ x_i = q_i - q_{0i} \]

At equilibrium:
\[ \frac{\partial U}{\partial q_i} \bigg|_{q_i=q_{0i}} = 0 \]

\[ U = U_0 + \frac{1}{2} \sum_{ij} K_{ij} x_i x_j + \ldots \]

Neglecting higher terms:

Kinetic energy:
\[ T = \frac{1}{2} \sum \dot{q_i} \dot{q_i} \]

To second order in \( x \), we can take:
\[ a_{ij} = \text{constant} \equiv m_{ij} \]

\[ m_{ij} = m_i \]

\[ m_{ij} > 0 \quad \text{because kinetic energy is positive} \]

Lagrangian:
\[ L = \frac{1}{2} \sum_{ij} m_{ij} \dot{x}_i \dot{x}_j - K_{ij} x_i x_j \]

\[ EL \quad m_{ij} \ddot{x}_j - K_{ij} x_j = 0 \]

This is a differential equation with constant coefficients and can be solved by substituting \( x_j = A e^{i \omega t} \).
\[ (-\omega^2 m_{ik} + \kappa i k) A_k = 0 \]

Non-zero solutions exist if the determinant vanishes:
\[ \det (-\omega^2 m_{ij} + \kappa i j) A_j^{(k)} = 0 \]

Solutions \( \omega_k^2 \) (characteristic equation)

Eigenvalues \( \omega_k^2 \).

\[ (-\omega_k^2 m_{ij} + \kappa i j) A_j^{(k)} = 0 \]

\[ \Rightarrow \quad A_i^{(k)} \omega_k^2 m_{ij} A_j^{(k)} = A_i^{(k)} \kappa i j A_j^{(k)} \]

Subtract complex conjugate and use that \( \kappa i j \) is real and symmetric,
\[ \Rightarrow \quad \left( \omega_k^2 - \omega_k^{2*} \right) \left( A_i^{(k)} \omega_k^2 m_{ij} A_j^{(k)} \right) = 0 \]

\( \omega_k^2 - \omega_k^{2*} = 0 \) because \( m_{ij} \) is positive definite,

\[ \Rightarrow \quad \omega_k = \omega_k^{*2} = 1 \quad \Rightarrow \quad \omega_k^2 \in \mathbb{R} \]
Let us now combine the equations as

\[ (1) \quad A_i^x (\omega - \omega_{e} \delta_{ij} + \kappa) A_j^{\mu(x)} = 0 \]
\[ \quad \hat{A}_i^x (-\omega_{e} \delta_{ij} + \kappa) A_j^{\mu(x)} = 0 \]
\[ \quad \hat{\theta}_j^x (-\omega_{e} \delta_{ij} + \kappa) \dot{A}_i^{\mu(x)} \]
\[ \quad \kappa_{ij} = \kappa_{ji} \]

\[ (2) \quad \omega_{c} (\omega_{c} - \omega_{e}) (A_i^{\kappa} \delta_{ij} A_j^{\kappa}) = 0 \]
\[ \quad \delta_{ij} = \frac{1}{A_i^{\kappa} \delta_{ij} A_j^{\kappa}} = 0 \]

We can always normalize eigenvectors such that

\[ \hat{A}_i^{\mu(x)} \delta_{ij} A_j^{\mu(x)} = \delta \delta_{ij} \]

Expand \( x_i \) in eigenvectors

\[ x_i^x = \sum_{\kappa} A_i^{\kappa} Q_\kappa \]
\[ x_i^{\kappa} = x_i \]

\[ \frac{1}{2} \dot{x}_i^{\kappa} \delta_{ij} = \frac{1}{2} \sum A_i^{\kappa} \delta_{ij} A_j^{\kappa} \frac{\omega_{c} Q_\kappa Q_\mu}{\omega_{c} \delta_{ij} A_j^{\kappa}} \]
\[ = \frac{1}{2} \sum A_i^{\kappa} Q_\kappa Q_\mu \delta_{ij} \]
\[ = L = \frac{1}{2} \sum A_i^{\kappa} \dot{Q}_\kappa \dot{Q}_\mu - \frac{1}{2} \omega_{c} \dot{Q}_\kappa \dot{Q}_\mu \]

Time dependence of normal modes

\[ Q_\kappa(t) = Q_\kappa(0) e^{i \omega_{c} t} \]
6b) Coupled pendulum

\[ T = \frac{1}{2} m \dot{\theta}_1^2 + \frac{1}{2} m \dot{\theta}_2^2 \]

\[ U = mgL(1 - \cos \theta_1 + 1 - \cos \theta_2) + \frac{1}{2} k \left( \frac{\dot{\theta}_1}{1 - \cos \theta_1} + \frac{\dot{\theta}_2}{1 - \cos \theta_2} \right)^2 \]

Choose dimensionless variables:

\[ \lambda = L = \frac{1}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \left[ \frac{1}{2} (\theta_1^2 + \theta_2^2) + \frac{1}{8} (\theta_1 - \theta_2)^2 \right] \]

Equilibrium at \( \theta_1 = \theta_2 = 0 \)

\[ m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 + \lambda & -\lambda \\ -\lambda & 1 + \lambda \end{pmatrix} \]

Eigenvalue equation:

\[ \det (\lambda - \lambda \mathbf{m}) = 0 \]

\[ (1 + \lambda - \lambda)^2 - \lambda^2 = 0 \]

\[ \lambda_1 = 1, \quad \lambda_2 = 1 + 2 \lambda \]

Eigenvalues: \( \lambda = \lambda_1, \lambda_2 \)

Eigenvalues:

\[ \lambda = 1, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \lambda = 1 + 2 \lambda, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Expansion in normal modes:

\[ \mathbf{\Theta}_i = \mathbf{A}_i^0 \mathbf{Q}_1 + \mathbf{A}_i^{(1)} \mathbf{Q}_1 + \mathbf{A}_i^{(2)} \mathbf{Q}_2 + \mathbf{A}_i^{(3)} \mathbf{Q}_3 \]

\[ \Theta_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \Theta_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]
\[ T = \frac{1}{2} (\dot{\theta}_1 + \dot{\theta}_2) = \frac{1}{2} \dot{\theta}_1^2 + \frac{1}{2} \dot{\theta}_2^2 \]

Eqs. of motion

\[ \ddot{\theta}_1 = -\omega_1 \]

\[ \ddot{\theta}_2 = -(1 + 2\omega) \theta_2 \]

\[ \theta_1 = a_1 \cos t + b_1 \sin t \]

\[ \theta_2 = a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \]

\[ \omega_2 = \sqrt{1 + 2\omega} \]

\[ \Theta_1 = \frac{\theta_1 + \theta_2}{\sqrt{2}} \]

\[ \Theta_2 = \frac{\theta_1 - \theta_2}{\sqrt{2}} \]

\( a_1, a_2, b_1, b_2 \) are determined by the initial conditions

6c) Zero modes

What is the meaning of zero frequencies?

\[ \omega = 0 \]

\[ \Theta_k = 0 \]

\[ \Theta_k = a + b \]

This is motion with constant velocity.

All atoms move in the same direction.
6.1) Perturbation theory for the anharmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{1}{2} \omega_0^2 x^2 + \frac{1}{4} \epsilon x^4 \]

\[ m = 1 \]

\[ \dot{x}^0 + \omega_0^2 x + \epsilon x^3 = 0 \]

perturbative expansion \( x(t) = x^0(t) + \epsilon x^1(t) + \epsilon^2 x^2(t) + \cdots \)

\[ \dot{x}^0 + \epsilon \dot{x}^1 + \epsilon^2 \dot{x}^2 + \frac{d}{dt}(x^0 + \epsilon x^1 + \epsilon^2 x^2) \]

\[ + \epsilon (x^0 + 3 \epsilon x^0 x^1 + \frac{1}{2} \epsilon^2 x^0 x^1 x^1) + \mathcal{O}(\epsilon^3) \]

\[ \mathcal{O}(\epsilon^3) \quad \dot{x}^2 = -\frac{\omega_0^2}{\epsilon} x^0 \]

\[ \mathcal{O}(\epsilon) \quad \dot{x}^1 + \omega_0^2 x^1 + x^0 = 0 \]

initial conditions:

\[ x(0) = a \quad \forall \epsilon \quad \dot{x}(0) = 0 \]

\[ x^0(0) = a \]

\[ \dot{x}^0(0) = 0 \]

\[ x^1(t) + \omega_0^2 x^1 = -\frac{3}{2} a \cos \omega_0 t \]

Solution:

\[ x(t) = -\frac{3}{2} \omega_0 \left( 3 \omega_0 t \sin \omega_0 t + \frac{1}{2} \cos \omega_0 t \right) \]

\[ -\frac{1}{4} \cos 3 \omega_0 t \]

diverges for \( t \to \infty \)

Reason for divergence: true resonance frequency is \( \omega_0 + \omega_0 \)

\[ \Box \]
5. If we expand \( \cos((\omega_0 + \epsilon \omega_1) t) \) using small angle approximations near \( \epsilon = 0 \) and \( \omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots \),

\[
x(t) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots
\]

Since \( x_0 = x_0(\omega_0 t) \), we have

\[
\dot{x}_0 = \omega_0 \dot{x}_0 \quad \Rightarrow \quad \dot{x}_0 = \omega_0 \frac{dx_0}{dt} = \omega_0^2 x_0
\]

Thus,

\[
\ddot{x}_0 + \omega_0^2 x_0 = 0
\]

For \( x_0 \), we have

\[
\dddot{x}_0 + \omega_0^2 x_0 = 0
\]

Assuming a solution of the form \( x_0 = a \cos \omega t \),

\[
\omega_0^2 (x_1'' + x_1) = 2 \omega_0 \omega_1 a \cos \omega t - \frac{3}{2} a^3 \cos^3 \omega t - \frac{3}{8} a^3 \cos 3\omega t
\]

We can choose \( \omega_1 \) such that the dangerous term \( \cos \omega t \) vanishes. For \( \omega_1 = \frac{3}{2} \omega_0 \),

\[
\omega_0^2 (x_1'' + x_1) = 2 \omega_0 \omega_1 a \cos \omega t - \frac{3}{2} a^3 \cos^3 \omega t - \frac{3}{8} a^3 \cos 3\omega t
\]
\text{Ansatz: } x_1 = a_1 \cos \tau + a_2 \cos 3 \tau

= \frac{a_2}{2} \left( -a_1 \cos \tau - 3a_2 \cos 3 \tau \right) + \frac{a_1}{2} \left( a_2 \cos \tau + a_2 \cos 3 \tau \right)

= \frac{a_2}{2} \cos 3 \tau

\Rightarrow a_2 = \frac{a_1}{32 \omega_0^2}

= x_1(\tau) = a_1 \cos \tau + \frac{a_1}{32 \omega_0^2} \cos 3 \tau

x_1(0) = 0 \Rightarrow a_1 = -\frac{a_1}{32 \omega_0^2}

= x_1(\tau) = \frac{a_1}{32 \omega_0^2} \left( -\cos \tau + \cos 3 \tau \right)

there are no infinities !!!

dangerous resonances occur frequently for perturbation theory in classical mechanics.
(a) Hamilton's equations

**Phase space**: Use \( q_i, p_k \) as coordinates

\[
\frac{\partial L}{\partial t_k} \quad \text{momenta} \quad p_k = \frac{\partial L}{\partial \dot{q}_k}
\]

Then, the EL equations which are of 2nd order can be written as first-order equations.

Hamiltonian \( H = \sum_{k=1}^{n} p_i q_i - L \)

\[
H = \begin{pmatrix} q_1, \ldots, q_n, p_1, \ldots, p_n, t \end{pmatrix}
\]

\[
\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial \dot{q}_i} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i
\]

\[
\frac{\partial H}{\partial p_i} = \dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial}{\partial p_i} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + p_i \frac{\partial}{\partial p_i} \frac{\partial L}{\partial q_i}
\]

\[
= q_i
\]

\[
\frac{dH}{dt} = \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial}{\partial q_i} \frac{\partial H}{\partial \dot{q}_i} = 0
\]

Energy is conserved if the Lagrangian is invariant under time translations.
Hamilton's equations

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$
$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$

If the kinetic energy is quadratic in the velocities, then

$$L = \frac{1}{2} \sum g_{ij} \dot{q}_i \dot{q}_j - U \Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i} = g_{ij} \dot{q}_j$$

$$H = p_i \dot{q}_i - L = g_{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} g_{ij} \dot{q}_j \dot{q}_j + U$$

$$= T + U$$

Sometimes it is useful to transform only part of the coordinates

i.e. $q, \dot{q}, \xi, \dot{\xi} \rightarrow q, p, \xi, \dot{\xi}$

this leads to the so called Routhian
Poisson brackets (section 9.5)

Let us consider a function on phase space $f(q, p, t)$.

Then

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p} + \frac{\partial f}{\partial t}$$

$$= \frac{df}{dt} = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p} + \frac{\partial f}{\partial t} + \left[ f, H \right]$$

$$\left[ f, H \right] = \sum_k \left( \frac{\partial H}{\partial p_k} \frac{\partial f}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

This is the classical limit of a commutator.

If $f$ is an integral of motion then $\frac{df}{dt} = 0$ then

$$\frac{df}{dt} + \left[ f, H \right] = 0$$

If $\frac{df}{dt} = 0$ as well then $\left[ H, f \right] = 0$

Generalized Poisson bracket

$$\left[ q_k, f \right] = \sum_k \left[ \frac{\partial f}{\partial q_k} \frac{\partial f}{\partial q_k} - \frac{\partial f}{\partial p_k} \frac{\partial f}{\partial p_k} \right]$$
Some properties of Poisson brackets

\[[ F, g ] = - [ g, F ] \]
\[[ F, c ] = 0 \]
\[[ F, \phi ] = c \text{ constant} \]
\[[ F + F_2, g ] = [ F_1, g ] + [ F_2, g ] \]
\[[ F_1, F_2, g ] = F_1 [ F_2, g ] + F_2 [ F_1, g ] \]

\[[ F, \phi \chi \frac{\partial}{\partial \phi} ] = -\frac{2 \chi}{\hbar} \]
\[[ F, \phi \frac{\partial}{\partial \phi} ] = \frac{2 F}{\partial \phi} \]
\[[ F_i, \phi \kappa ] = 0 \]
\[[ \phi_i, \phi \kappa ] = 0 \]
\[[ F, \phi \kappa ] = -8i \chi \]

Jacobi identity

\[[ F, [ g, h ] ] + [ g, [ h, f ] ] + [ h, [ f, g ] ] = 0 \]
\[[ F, [ g, h ] ] = \frac{2 F}{\partial \phi} \left( \frac{\partial g_2 h}{\partial \phi} - \frac{\partial g h_2}{\partial \phi} \right) + \frac{2 F}{\partial \phi \kappa} \left( \frac{\partial h \phi_2}{\partial \phi \kappa} - \frac{\partial h_\phi \phi_2}{\partial \phi \kappa} \right) + \frac{2 F}{\partial \kappa \phi} \left( \frac{\partial h \phi}{\partial \kappa \phi} - \frac{\partial h_\phi \phi}{\partial \kappa \phi} \right)
+ \text{cyclic} \]

\begin{align*}
&\frac{2 F}{\partial \phi} \left( \frac{\partial g_2 h}{\partial \phi} - \frac{\partial g h_2}{\partial \phi} \right) + \frac{2 F}{\partial \phi \kappa} \left( \frac{\partial h \phi_2}{\partial \phi \kappa} - \frac{\partial h_\phi \phi_2}{\partial \phi \kappa} \right) + \frac{2 F}{\partial \kappa \phi} \left( \frac{\partial h \phi}{\partial \kappa \phi} - \frac{\partial h_\phi \phi}{\partial \kappa \phi} \right) \\
&+ \text{cyclic} = 0
\end{align*}
Poisson theorem

\[
\frac{dF}{dt} = 0 \quad \frac{dG}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt} [F, G] = 0
\]

If both \( F \) and \( G \) have no explicit time dependence,

Use the Jacobi identity with \( h = H \)

\[
\]

\[
\Rightarrow \frac{d}{dt} [F, G] = 0
\]


\[ \textbf{Hamilton's principle} \]

\[
\int_{t_1}^{t_2} q(t) \quad s = s(t) \quad dt
\]

For fixed initial point, we consider the action as a function of the end point.

\[
\delta s = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt
\]

\[
= \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} \delta q_i - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] dt + \frac{\partial L}{\partial q_i} \delta q_i \bigg|_{t_1}^{t_2}
\]

\[
= 0 \quad \forall \delta q_i \in \mathcal{L}
\]

\[
\frac{\delta s}{\delta q(t_2)} = p(t_2)
\]

We can also consider \( s \) as an explicit function of time by taking \( t_1 \) fixed and \( t_2 = t \).

\[
\int \frac{\delta s}{\delta t} dt = \int (\mathbf{p} \dot{q} - H) dt
\]

\[
\Rightarrow \int (\frac{\delta s}{\delta q_i} \dot{q}_i + \frac{\delta s}{\delta \dot{q}_i} \ddot{q}_i) dt \quad \Rightarrow \quad H = -\frac{\delta s}{\delta q_i}
\]

\[
\Rightarrow \delta s = \int (\mathbf{p}_i \ddot{q}_i - H) dt
\]
\[ S = S_{as} = \int \sum \pi_i \, dq_i = 1 + dt \]

We now vary \( \pi_i \) and \( q_i \) independently.

\[ \delta S = \int \left[ \delta \pi_i \, dq_i + \pi_i \, d \delta q_i - \frac{\partial H}{\partial q_i} \frac{d \delta q_i}{dt} - \frac{\partial H}{\partial \pi_i} \right] \, dt \]

\[ \delta q_i = 0 \quad \text{at endpoint} \]

**Equate coefficients of** \( \delta q_i \) and \( \delta \pi_i \).

\[ \delta \pi_i = 0 \quad \text{at endpoint} \]

\[ \begin{align*}
\delta q_i & - \frac{\partial H}{\partial \pi_i} \, dt = 0 \quad \Rightarrow \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial \pi_i} \\
- \delta \pi_i & - \frac{\partial H}{\partial q_i} \, dt = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial q_i} = - \frac{dq_i}{dt}
\end{align*} \]

**Hamilton's Principle**
8a) Canonical transformations

Generalized coordinate transformations

\[ q_i \rightarrow q_i (Q_i, t) \]

Formal appearance of EL remain the same

\[ L(q_i) \rightarrow L'(Q_i) \]

\[ \frac{d}{dt} \frac{\partial L'}{\partial \dot{Q}_i} - \frac{\partial L'}{\partial Q_i} = 0 \]

Also Hamilton's equations remain the same

\[ p_i = \frac{\partial H}{\partial \dot{Q}_i} \Rightarrow \dot{p}_i = \frac{\partial H'}{\partial \dot{Q}_i} = \frac{\partial H}{\partial \dot{Q}_i} \frac{\partial L'}{\partial \dot{Q}_i} = \frac{\partial L'}{\partial Q_i} \]

Because \( p \) and \( q \) are independent, we have more freedom to transform \( p \) and \( q \) while keeping the structure of the Hamilton equations:

\[ q_i \rightarrow Q_i (p_i, q_i, t) \]

\[ p_i \rightarrow p_i (Q_i, q_i, t) \]

are canonical if

\[ p_i = -\frac{\partial H'}{\partial Q_i}, \quad Q_i = \frac{\partial H'}{\partial p_i} \]

H-equations for the new variables can be written as

\[ \delta \int (p_i \dot{q}_i - H' dt) = 0 \]

In terms of the old variables, we have

\[ \delta \int \left( \sum p_i \dot{q}_i - H dt \right) = 0 \]

H-eq. for the new variables is satisfied if

\[ p_i \dot{q}_i - H' dt - \left[ p_i dq_i - H dt \right] = 0 \text{ (final convention)} \]
$F$ is called a generating function

\[ p_i = \frac{\partial F}{\partial q_i} \quad p_i = -\frac{\partial F}{\partial q_i} \quad H' = H + \frac{\partial F}{\partial t} \]

Example

\[ F = \sum q_i q_i \]

\[ p_i = q_i \quad p_i = -q_i \]

Moments and coordinates are interchanged.

Alternative canonical transformations can be obtained by Legendre transformations.

\[ \phi = F + \sum p_i q_i \]

\[ d\phi = p_i dq_i - q_i H' dt - H dt + p_i dq_i + dp_i q_i \]

\[ p_i = \frac{\partial \phi}{\partial q_i} \quad q_i = \frac{\partial \phi}{\partial p_i} \quad H' = H + \frac{\partial \phi}{\partial t} \]

\[ \phi = \phi(q, p, t) \]
Invariance of Poisson brackets under covariant canonical transformations

\[ q_a p_c \rightarrow q_a' p_c' \]

\[ [f, g] = \frac{\partial f}{\partial q_a} \frac{\partial g}{\partial p_c} - \frac{\partial f}{\partial p_c} \frac{\partial g}{\partial q_a} \]

\[ = \left( \frac{\partial f}{\partial q_a} \frac{\partial q_a}{\partial q_c} + \frac{\partial f}{\partial p_c} \frac{\partial p_c}{\partial q_c} \right) \left( \frac{\partial g}{\partial p_c} \frac{\partial q_c}{\partial p_a} + \frac{\partial g}{\partial q_a} \frac{\partial q_a}{\partial p_a} \right) \]

\[- \left( \frac{\partial f}{\partial p_c} \frac{\partial q_c}{\partial q_a} + \frac{\partial f}{\partial q_a} \frac{\partial q_a}{\partial p_c} \right) \left( \frac{\partial g}{\partial q_a} \frac{\partial p_a}{\partial q_c} + \frac{\partial g}{\partial p_c} \frac{\partial p_c}{\partial q_a} \right) \]

\[ = \frac{\partial f}{\partial q_a} \frac{\partial g}{\partial q_c} + \frac{\partial f}{\partial p_c} \frac{\partial g}{\partial p_a} \]

we first look at the case of only one set \( q, r \) and \( \omega, p \) then \([\omega, p] = 0\)

\[ [f, g] = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \]

generating function \( F = \frac{p}{\partial q} \) \( F = FC(q, p) \)

\[ \frac{\partial F}{\partial q} = -\frac{\partial^2 F}{\partial q \partial p} \frac{\partial q}{\partial p} - \frac{\partial^2 F}{\partial p^2} \frac{\partial p}{\partial q} = 0 \]

\[ \frac{\partial F}{\partial p} = -\frac{\partial^2 F}{\partial q \partial p} \frac{\partial p}{\partial q} - \frac{\partial^2 F}{\partial p^2} \frac{\partial q}{\partial p} = 0 \]
\[ L = \frac{1}{2} \sum p_i q_i - \sum \frac{\partial F}{\partial q_i} \frac{\partial F}{\partial p_i} \]

\[ L = \frac{1}{2} \sum \frac{\partial^2 F}{\partial q_i \partial p_i} \frac{\partial q_i}{\partial p_j} \frac{\partial p_i}{\partial q_j} \]

To prove the general case we first show that Lagrange brackets are invariant under canonical transformations.

Lagrange brackets are defined as

\[ (L, M) = \sum (\frac{\partial q_i}{\partial L} \frac{\partial p_i}{\partial M} - \frac{\partial p_i}{\partial L} \frac{\partial q_i}{\partial M}) \]

We now use the generating function

\[ F(q, p) \]

Then

\[ p_i = \frac{\partial F}{\partial q_i}, \quad q_i = \frac{\partial F}{\partial p_i} \]

And

\[ \frac{\partial p_i}{\partial q_j} = \frac{\partial^2 F}{\partial q_i \partial q_j} + \frac{\partial^2 F}{\partial q_i \partial p_j}, \quad \frac{\partial q_i}{\partial p_j} = \frac{\partial^2 F}{\partial q_i \partial p_j} \]

Same for \( \frac{\partial p_i}{\partial q_j} \)

\[ (L, M) = \frac{\partial F}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial F}{\partial q_i} \frac{\partial F}{\partial p_i} = 0 \quad \text{symmetric} \]

\[ + \frac{\partial^2 F}{\partial q_i \partial p_i} \left( \frac{\partial q_i}{\partial q_j} - \frac{\partial q_i}{\partial p_j} \right) \]

\[ + \frac{\partial^2 F}{\partial q_i \partial p_i} \left( \frac{\partial p_i}{\partial q_j} - \frac{\partial p_i}{\partial p_j} \right) = \text{additive} \quad \text{cyclic} \]
\[
\begin{align*}
\text{LS} = \frac{2}{Q_p} \left( \frac{\partial F}{\partial u^p} \frac{\partial \epsilon}{\partial \epsilon} - \frac{\partial F}{\partial \epsilon} \frac{\partial \epsilon}{\partial u^p} \right) \\
= \left[ u^i, u^j \right]_{Q_p}
\end{align*}
\]

Lagrange brackets are invariant under canonical transformations.

Next we show that the Lagrange and Poisson brackets are related by

\[
\sum_{i,j} \left[ \epsilon_i, u^j \right] \left[ \epsilon_i, u^j \right] = \delta_{ij}
\]

for a set \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) of independent functions of \( q_1, q_2, \ldots, q_n \).

\[
\sum_{i,j} \left( \frac{\partial^2 F}{\partial q^i \partial q^j} - \frac{\partial^2 F}{\partial q^j \partial q^i} \right)
\]

Use the chain rule

\[
\frac{\partial \epsilon_i}{\partial q^m} = \frac{\partial \epsilon_i}{\partial u^p} \frac{\partial u^p}{\partial q^m}
\]

\[
= \left[ u^i, u^j \right] = \delta_{ij}
\]
Finally, we choose \( u_i = q_i \), \( u_j = p_j \), \( \delta_{ij} = 0 \)

\[
\sum_{k} \left( \{ q_k, q_i \} [ q_k, p_j ] + \{ p_k, q_i \} [ p_k, p_j ] \right) = 0
\]

\[
\Rightarrow \{ p_k, q_i \} [ p_k, p_j ] = \{ q_k, q_i \} [ q_k, q_j ] = \delta_{ij}
\]

\[
\Rightarrow \{ p_k, q_i \} [ p_k, p_j ] = \delta_{ij}
\]

The Poisson bracket is invariant under canonical transformations.

\[
\Rightarrow [ p_k, q_j ] = -\delta_{kj}
\]

Invariance of the Poisson bracket for an arbitrary function of \( q_1 \) and \( q_2 \) then follows by induction.
Symplectic approach to canonical transformation

\[ Q_i = Q_i(q, p) \quad P_i = P_i(q, p) \]

\[ \frac{\partial Q_i}{\partial p_j} \dot{q}_j + \frac{\partial Q_i}{\partial q_j} \dot{p}_j = \frac{\partial H}{\partial p_i} \]

Inverse transformation \[ q_j = q_j(q, p) \]
\[ p_j = p_j(q, p) \]

Then we can write \( H \) as a function of \( q, p \)

\[ \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial p_i} + \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial p_i} \]

\[ Q_i = \frac{\partial H}{\partial p_i} \quad \frac{\partial p_i}{\partial q_j} = -\frac{\partial Q_j}{\partial q_i} \]
\[ \frac{\partial q_j}{\partial p_i} = \frac{\partial p_i}{\partial q_j} \]

From \( p_i \) we obtain

\[ \left( \frac{\partial p_i}{\partial q_j} \right)_{q_i} = -\frac{\partial p_i}{\partial q_i} \]
\[ \frac{\partial p_i}{\partial q_j} = \frac{\partial q_j}{\partial p_i} \]
\[ \frac{\partial q_j}{\partial p_i} = \frac{\partial p_i}{\partial q_j} \]

\[ \frac{\partial p_i}{\partial q_i} = \frac{\partial q_i}{\partial p_i} \]
symplectic matrix \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

\[ k = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \]

\[ \dot{k} = \text{Hessian} \quad \frac{dk}{dt} = J \frac{\partial H}{\partial q} \]

\[ y = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \\ P_1 \\ \vdots \\ P_n \end{pmatrix} \quad y = y(h) \]

\[ \dot{y} = M \dot{h} = M J \frac{\partial H}{\partial q} \]

\[ M_{ij} = \frac{\partial y_i}{\partial q_j} \]

\[ \frac{\partial H}{\partial q} = \sum \frac{\partial H}{\partial q_j} = 0 \Rightarrow \dot{y} = M J M \frac{\partial H}{\partial y} \]

\( \tilde{M}_{ij} = \frac{\partial y_j}{\partial q_i} \quad M \text{ is transpose of } M \)

transformation is canonical if \( M^T \tilde{M} = J \)

\( \text{symplectic condition} \)
\[ \frac{\partial y_i}{\partial q_j} \frac{\partial p_j}{\partial \dot{q}_k} = \delta_{ik} \]

\[ \frac{\partial y_i}{\partial q_j} \frac{\partial p_j}{\partial \dot{p}_k} - \frac{\partial y_i}{\partial \dot{q}_j} \frac{\partial p_j}{\partial \dot{p}_k} = \delta_{ik} \]

If the Poisson brackets are invariant, then the transformation is canonical.

**P1)** Liouville theorem

\[
\begin{align*}
\text{Volume } d\Gamma &= dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n \\
\text{Canonical transformation } &d\Gamma' = dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n \\
\text{Liouville theorem } &d\Gamma = d\Gamma' \\
\end{align*}
\]

To prove this we have to show that the Jacobian \( (q, p) \rightarrow (q', p') \) is equal to 1.
\[ \mathcal{Z} = \frac{\mathcal{Z}(q_1, \ldots, q_n, p_1, \ldots, p_n)}{\mathcal{Z}(q_1, \ldots, q_n, p_1, \ldots, p_n)} \]

\[ = \mathcal{Z}(q_1, \ldots, q_n, p_1, \ldots, p_n) \frac{\partial}{\partial q_i} \mathcal{Z}(q_1, \ldots, q_n, p_1, \ldots, p_n) \]

\[ \quad \det \frac{\partial q_i}{\partial q_i} \]

\[ \Rightarrow \eta = 1 \]

Generating function of canonical transformation

\[ \phi(q, p) \]

\[ q_i = \frac{\partial \phi}{\partial p_i}, \quad p_i = \frac{\partial \phi}{\partial q_i} \]

\[ \Rightarrow \eta = 1 \]

Time evolution can be viewed as a canonical transformation

\[ q^k_t, p^k_t \rightarrow q^k_{t+\tau}, p^k_{t+\tau} \]

\[ q^k_{t+\tau} = q^k_t (q^k_t, p^k_t) \]

\[ S_t = \sum_{t=0}^{\tau} (p dq - H dt) \]

\[ S^*_{t+\tau} - S_t = \sum_{t=0}^{\tau} (p dq - H dt) dt \]

\( \Rightarrow S^*_{t+\tau} - S_t \) is generating function of canonical transformation.
= time evolution can be viewed as a series of canonical transformations

\[ \sum \frac{d\xi}{dt} = \sum \frac{d\xi}{dt+e} \]

For any region of phase space

\[ \sum d\xi d\eta \]

\[ F = F(q, p, t) \]

\[ q_n = \frac{\partial F}{\partial p_n} , \quad p_n = \frac{\partial F}{\partial q_n} \]

\[ \sum (\frac{\partial q_n}{\partial q_k} dq_k + \frac{\partial q_n}{\partial \eta_k} d\eta_k) (\frac{\partial p_k}{\partial q_n} dq_n + \frac{\partial p_k}{\partial \eta_n} d\eta_n) \]

\[ dq_k dq_m = -L_{mn} dq_k \]

\[ dq_k d\eta_m = -L_{mn} d\eta_m \]

etc.

\[ \sum d\xi d\eta = \sum (\frac{\partial q_n}{\partial q_k} dq_k + \frac{\partial q_n}{\partial \eta_k} d\eta_k) (\frac{\partial p_k}{\partial q_n} dq_n + \frac{\partial p_k}{\partial \eta_n} d\eta_n) \]

\[ \left( \frac{\partial p_k}{\partial p_n} - \frac{\partial q_n}{\partial \eta_n} \right) dq_k d\eta_n \]

Lagrangian is invariant \( \Rightarrow \) under time evolution
Poincare section

2 dof of $x_1, x_2, x_3, x_4$

$E$ conserved $\Rightarrow$ trajectory in $\mathbb{R}^d$

Poincare section is the collection of section points with $x_1 > 0$

2 conditions on $y$ variables $E, x_4 = 0$

$\Rightarrow$ points cover a 2d area

For two conserved quantities on $x_1, x_2, x_3$ with $x_3 = 0$, we have 3 conditions on $y$ variables $\Rightarrow$ section points are on a curve.

Example

h.o. in $\mathbb{R}^d$

2 conserved quantities

$\frac{1}{2}x_1^2 + \omega_1^{-2}x_2^2 = E_1$

$\frac{1}{2}x_3^2 + \omega_3^{-2}x_4^2 = E_2$

if $\frac{\omega_1}{\omega_3} \in \mathbb{Q}$, then we have only a finite number of section points.
Lagrangian and Hamiltonian for electromagnetic field

Lorentz force \( F^a = e E^a + \frac{1}{c} \epsilon^{abc} \dot{u}_b \dot{u}_c B^a \)

Eqs. of motion \( m \ddot{x}^a = F^a \)

We express \( E \) and \( B \) fields in terms of vector potential \( E_\alpha = -\partial_\alpha \phi - \partial^\tau A_\tau \)

\( B_\alpha = \epsilon_{\alpha\beta\gamma} \partial_\beta A_\gamma \)

\( m \ddot{x}_\mu = -e \partial_\mu \phi - e \partial^\tau A_\tau + e \epsilon^{\tau\mu\nu\delta} \partial_\nu \dot{x}_\delta A_\tau \)

\( = -e \partial_\mu \phi - e \partial^\tau A_\tau + e (\partial_\mu \partial_\tau - \partial_\tau \partial_\mu) \dot{x}_\tau A_\mu \)

\( = \frac{d}{dt} \left( m \dot{x}_\mu + e A_\mu \right) = \partial_\mu \left( e \partial_\mu A_\mu - e \phi \right) \)

Total derivative

\( \frac{d}{dt} \left( m \dot{x}_\mu + e A_\mu \right) = \partial_\mu \left( e \partial_\mu A_\mu - e \phi \right) \)

\( L = \frac{1}{2} m \dot{x}_\mu \dot{x}^\mu + e A_\mu \dot{x}^\mu - e \phi \)

\( \frac{d}{dt} (e A_\mu) = e \partial_\mu A_\tau + e \delta_\mu A_\mu \dot{x}_\tau \)

\( p_\mu = \frac{dL}{dx_\mu} = m \dot{x}_\mu + e A_\mu \)

Momentum is not equal to \( \dot{x}_\mu \)
\[ H = p \cdot \dot{a} - L = P \cdot \left( \frac{p - e \phi}{m} \right) \]

\[- \frac{1}{2} m \left( \frac{p - e \phi}{m} \right)^2 + e \phi \frac{p - e \phi}{m} \]

\[ = \frac{1}{2} m \left( \frac{p - e \phi}{m} \right)^2 + e \phi \]

**gauge invariance**

\[ \theta \rightarrow \theta + \delta \lambda \]

\[ \phi \rightarrow \phi - \partial t \lambda \]

Do not change \( \mathbb{E} \) and \( B \)

\[ L = L + e \epsilon \phi \lambda + \frac{\epsilon}{2} \mathcal{L} \]

\( \Rightarrow \) eqs. of motion are the same

**Example**  \[ \text{B-constant} \]

\[ B_{\gamma} = e \chi_{\alpha} \partial \beta \]

\[ \theta_{\alpha} = -\frac{\partial}{\partial y} \]

\[ \alpha_{\gamma} = \frac{B}{\epsilon} \lambda \]

\[ \alpha_{\alpha} = B y \]

\[ \alpha_{\gamma} = 0 \]

\[ a_{\alpha} = a_{\alpha} - \partial \lambda \left( \frac{B}{\epsilon} y \right) \]
Hamilton-Jacobi Theory

\[ H = -\frac{\partial S}{\partial t} \quad \Rightarrow \quad \frac{\partial S}{\partial t} + H(q_i, \frac{\partial S}{\partial q_i}, t) = 0 \]

\[ p_i = \frac{\partial S}{\partial q_i} \]

This is the Hamilton-Jacobi equation for \( S(q_i, t) \)

Complete integral: a solution of the HJ eq that contains as many arbitrary constants as independent variables.

Time + coordinates \( \rightarrow \) \( S + 1 \) independent variable

Since \( S \) is only determined up to a constant, a complete integral is given by

\[ S = F(t, q_1, \ldots, q_n, \pi_1, \ldots, \pi_n) + A \]

Relation between complete integral and solution of the eqs. of motion

Use \( F \) as generating function for canonical transformation \( q_i, \pi_i \rightarrow q_i' \pi_i' \) new coordinates

\[ \pi_i = \frac{\partial F}{\partial q_i} \quad \pi_i = \frac{\partial F}{\partial q_i} \quad H = H + \frac{\partial F}{\partial t} = 0 \]

- 14
\[ \frac{d^2x}{dt^2} + \omega^2 x = 0 \]

Initial conditions:

\[ x(0) = 0, \quad \frac{dx}{dt}(0) = 0 \]

Solution:

\[ x(t) = \sin(\omega t \pm \phi) \]

Example: Harmonic Oscillator

\[ \beta_i = \frac{\partial^2}{\partial t^2} \]

Solve the Hamiltonian, \( H \).

Hamilton: \[ H = \frac{\beta_i}{2} + \frac{1}{2} \beta^2 + \frac{1}{2} p^2 = 0 \]

\[ \beta = \text{constant} \]

\[ \frac{dx}{dt} = \frac{\partial}{\partial \beta} \]

The Hamiltonian is constant in time.
Action--angle variables.

When we discussed Poisson brackets, we saw that if there is an additional constant of motion for $α$ of the form, the motion is on ellipses. Ellipse is specified by $J = α φ$. It makes sense to use $J$ as one variable and the angle around the ellipse as the other variable.

$J = α$ is a constant of motion.

Our construction will be that $J$ will be the momentum conjugate to $φ$.

\[ J = α = \frac{∂H}{∂φ} \Rightarrow H \text{ does not depend on } φ \]

\[ φ_0 = \frac{∂H}{∂J} = λJ \]

"constant \[ = J_0 = cτ t + Θ^2 \]

$⇒$ motion is on tori

What is the canonical transformation $(q, p) \rightarrow (α, J)$? Has to be canonical because $α, J$ should satisfy the Hamilton eqs of motion.
Action Angle Variables

\[ s = s_0 - Et \]
\[ s_0 = \sum_k \xi_k (q, Q) \]

\[ (q, p) \xrightarrow{\tilde{s}(q, Q)} (\phi, \tau) \]
\[ p_\phi = \frac{\partial \tilde{s}}{\partial q}, \quad p_\tau = \frac{\partial \tilde{s}}{\partial Q} \]

A solution of Hamilton Eq.
\[ p_\phi = \frac{\partial s_0}{\partial q}, \quad L_\phi = \frac{\partial s_0}{\partial Q} \]

What is \( \tilde{s}(q, Q) \)?

\( \phi_\alpha \) are angular variables \( \Rightarrow \int C_a d\phi_\alpha = 2\pi \)

\[ \int_C a d\phi_\alpha = \int_C a \frac{\partial \tilde{s}}{\partial q} = \int_C a \frac{\partial s_0}{\partial q} d\tilde{q}_{\phi_\alpha} \]

\[ = \frac{2}{\partial q^\alpha} \int_C p_\alpha d\tilde{q}_{\phi_\alpha} = 2\pi \frac{f_{\phi_\alpha}}{Cc} \]

This suggests that \( f_{\phi_\alpha} = \frac{1}{2\pi} \int_C p_\alpha d\tilde{q}_{\phi_\alpha} \)

assume that we have solved the Hamilton Eq.
then
\[ f_{\phi_\alpha} = \frac{1}{2\pi} \int_C \frac{\partial s_0}{\partial q} \] this gives \( f_{\phi_\alpha}(q) \)
The correct choice for $\tilde{S}$ is given by:

$$\tilde{S} = \sum \tilde{S}_a(q_a, Q(2))$$

**Proof:**

$$\frac{\partial \tilde{S}}{\partial q_a} = \tilde{S}_a(q_a, Q(2)) = 0$$

Angular variables:

$$\phi_a = \frac{\partial \tilde{S}}{\partial q_a}$$

We check that $\phi_a$ is indeed an angle:

$$\oint \phi_a \, dq_a = \oint \left( \frac{\partial \tilde{S}}{\partial q_a} \right) \, dq_a = \oint \frac{\partial}{\partial q_a} \tilde{S}_a \, dq_a = \oint \frac{\partial}{\partial q_a} \tilde{S}_a = \oint 0 = 0$$

**Example:** Particle moving on a circle

$$H = \frac{P_\theta}{2mR^2}$$

$$H = \frac{P_\theta}{2mR^2}$$

$$E = \left( \frac{\partial \tilde{S}}{\partial \theta} \right)^2 \frac{1}{2mR^2}$$

$$= \int \frac{\partial \tilde{S}}{\partial \theta} = \sqrt{2mE} \, R$$

$$\phi_\theta = \frac{\partial \tilde{S}}{\partial \theta} = \sqrt{2mE} \, R$$

$$\dot{\phi} = \frac{1}{i\hbar} \sum \dot{p}_a q_a = \frac{1}{i\hbar} \sum p_\theta d\theta = \sqrt{2mE} \, R = \frac{P_\theta}{\hbar}$$

$$E(\theta) = \frac{\dot{\phi}^2}{2mR^2}$$

$$\tilde{S} = S(\theta, Q(2))$$

$$\phi = \frac{\partial \tilde{S}}{\partial \theta} = \theta$$

New variables are $\theta$ and $\phi$.\[\square\]
Example: Particle moving on a circle

\[ H = \frac{p_0^2}{2mR^2} \]

H) eqn \[ E = \left( \frac{\partial S_0}{\partial \theta} \right)^2 \frac{1}{2mR^2} \]

\[ \Rightarrow \frac{\partial S_0}{\partial \theta} = \sqrt{2mE}R^2 \]

\[ S_0(\theta, E) = \sqrt{2mE}R^2 \theta \]

\[ p_0 = \frac{\partial S_0}{\partial \theta} = \sqrt{2mE}R^2 \]

\[ J = \frac{1}{2\pi} \int \frac{1}{\theta} \int p_0 \, d\theta = \sqrt{2mE}R \equiv P_0 \]

\[ \Rightarrow EJ = \frac{J^2}{2mE} \]

\[ S(\theta, J) = S(\theta, \omega_0 J) = S(\theta, \frac{J^2}{2mE}) \]

\[ \theta = \Theta, \quad \mathcal{F} = \frac{\partial S}{\partial \Theta} = 0 \]

\( \Theta, J \) are angle-action variables
### Poincaré Sections and Area-Preserving Maps

**Poincaré Map**

\[ q_n = G_1(q_{n-1}, p_{n-1}) \]
\[ p_n = G_2(q_{n-1}, p_{n-1}) \]

Map from \( q_{n-1}, p_{n-1} \) to \( q_n, p_n \)

**Area-Preserving if**

\[ dq_{n-1} \times dp_{n-1} = dq_n \times dp_n \]

\[ dq_n = \frac{\partial G_1}{\partial q_{n-1}} dq_{n-1} + \frac{\partial G_1}{\partial p_{n-1}} dp_{n-1} \]
\[ dp_n = \frac{\partial G_2}{\partial q_{n-1}} dq_{n-1} + \frac{\partial G_2}{\partial p_{n-1}} dp_{n-1} \]

\[ dq_n \times dp_n = (\frac{\partial G_2}{\partial q_{n-1}} \frac{\partial G_1}{\partial p_{n-1}} - \frac{\partial G_1}{\partial q_{n-1}} \frac{\partial G_2}{\partial p_{n-1}}) dq_{n-1} dp_{n-1} \]

**Poisson Bracket**

\[ \{ q_n, p_n \} = 1 \]

Because \( q_n, p_n \) are obtained from \( q_{n-1}, p_{n-1} \)

by Hamiltonian time evolution

**Poincaré map is area preserving**
Lyapunov exponents

\[ \begin{align*}
X_{n+1} &= f(X_n, Y_n) \\
Y_{n+1} &= g(X_n, Y_n)
\end{align*} \]

map

Linearize \( \delta X_{n+1} = A_n \delta X_n \)

\[ A_n = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix} \]

\[ \delta X_n = A_n \delta X_0 \]

Eigenvalues \( \Lambda_B \):

\[ B_n \begin{pmatrix} \lambda_1(n) \\ \lambda_2(n) \end{pmatrix} = \mu(n) \begin{pmatrix} \lambda_1(n) \\ \lambda_2(n) \end{pmatrix} \]

For \( n \) large:

\[ \lambda_1(n) \approx e^{\lambda_1} \\
\lambda_2(n) \approx e^{\lambda_2} \]

\( \lambda_1 \) and \( \lambda_2 \) are known as Lyapunov exponents

\[ \det A_n \delta X_n \times \delta Y_n = \det B_n \delta X_0 \times \delta Y_0 \]

area preserving map: \( \lambda_1 + \lambda_2 = 0 \)

dissipative map: \( \lambda_1 + \lambda_2 < 0 \)
Motion is chaotic if one of the Lyapunov exponents is larger than zero.

\[ \Delta x(t) = \Delta x_0 e^{\lambda t} \quad \text{or} \quad \Delta x = \Delta x_0 e^{\lambda t} \]

The distance between two points in phase space increases exponentially.

This means that there is an extreme sensitivity to initial conditions.

This phenomenon is known as deterministic chaos.

For integrable systems, the Lyapunov exponents are zero.

For chaotic systems there are less conserved quantities than dof.

For a chaotic system \( \Delta x(t) \) can only be computed if we work with exponential accuracy and we need initial conditions with arbitrary accuracy. Of course, this is not possible.
11c) The cat map

\[ x_{n+1} = (2x_n + y_n) \mod 1 \]
\[ y_{n+1} = (x_n + y_n) \mod 1 \]

The hatched area is transformed into the hatched area.

A cat gets chopped.
linearized cat map

\[
\begin{pmatrix}
X_{n+1} \\
Y_{n+1}
\end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_n \\
Y_n
\end{pmatrix} \equiv A \begin{pmatrix} X_n \\
Y_n
\end{pmatrix} \mod 1
\]

let \( A = 1 \) \( \equiv \) map is area preserving

eigenvalues \( (2-\lambda)(1-\lambda) - 1 = 0 \)

\[
\lambda^2 - 3\lambda + 1 = 0
\]

\[
\Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2}
\]

\( \lambda_1 \lambda_2 = 1 \)

\[
A = \begin{pmatrix} 5/2 & 1/2 \\ 1 & 1/2 \end{pmatrix}
\]

\[
A^2 = \begin{pmatrix} 5/2 & 1/2 \\ 1 & 1/2 \end{pmatrix}^2 = \begin{pmatrix} 12/5 & 2/5 \\ 2/5 & 12/5 \end{pmatrix}
\]

\[
A^n = \begin{pmatrix} F_{2n} & F_{2n-1} \\ F_{2n-1} & F_{2n} \end{pmatrix}
\]

\[
\begin{align*}
F_{n+1} &= F_n + F_{n-1} & \text{Fibonacci numbers} \\
F_0 &= F_1 = 1
\end{align*}
\]

Initial conditions that are rational result in periodic trajectory

\[
(x_0, y_0) = \left( \frac{r}{N}, \frac{q}{N} \right) \quad r, q \in \mathbb{Z}
\]

then the iterated points are on a lattice with spacing \( \frac{1}{N} \)

map is invertible \( \Rightarrow \) must return to starting point in at least \( N^2 \) steps
1.3) Kicked Rotor

The simplest systems that exhibit chaotic motion are not systems with 2 dof but systems with 1 dof that are kicked periodically.

\[ H_0 = \frac{1}{2} p^2 \]

Force applied at \( t = 0, T, 2T, 3T \)

Torque \( \tau = RF \sin \phi \)

No gravity

\[
\begin{align*}
\tau &= F L = \varepsilon \\
\frac{d\phi}{dt} &= \frac{\dot{\phi}}{2} \\
\phi(nT + \tau) &= \phi(nT - \tau) + \frac{\varepsilon}{L} \sin \phi_n \\
\phi &= \frac{\dot{\phi}}{\omega} = \frac{\dot{\theta}}{2} \Rightarrow \phi_{n+1} = \phi_n + \frac{\varepsilon}{L} \sin \phi_n \\
\text{choose units with } \frac{1}{T} = 1
\end{align*}
\]
\( \phi_{n+1} = (\phi_n + j_{n+1}) \mod 2\pi \)
\( j_{n+1} = e \sin \phi_n + j_n \)

This map is known as the standard map

\[(\phi_{n+1}, j_{n+1}) = 2e (\phi_n, j_n)\]

Some properties:
- \(2e(\phi + 2\pi, j) = 2e(\phi, j)\)
- Map does not change if we add \(2\pi\) to all \(j_n\)

Therefore we have to consider only initial conditions in \(\phi \in [0, 2\pi]\)

\( j \in [-\pi, \pi]\)

Fixed points
\( \phi = (\phi + 2\pi) \mod 2\pi \)
\( j = e \sin \phi + j \)

\( \phi = 0, \pi \)
\( j = 0 \)

Map for \(e = 0\)
\( \phi_{n+1} = (\phi_n + j_{n+1}) \mod 2\pi \)
\( j_{n+1} = j_n \)
11b) Classification of systems

Integrable systems: as many conserved quantities as dof

Algebraic systems: integrable systems with additional conserved quantities so that the eqs. of motion can be solved algebraically (e.g. Kepler, etc.)

Non-integrable systems: systems with fewer conserved quantities than dof.

Ergodic systems: almost all trajectories cover phase space uniformly. Very important for statistical physics because it allows us to replace a time average by an ensemble average.

\[ \lim_{t \to \infty} \frac{1}{t} \langle A(t) \rangle = \frac{1}{V(\text{tot})} \int A \, dV \]

Mixing systems

\[ \frac{\sigma(B \cap T^k A)}{\sigma(B)} = \frac{\sigma(A)}{\sigma(\text{tot})} \]

Phase space: total phase volume

Space volume
V-systems: trajectories from almost all nearby points diverge exponentially on average.

Borel–Kάroli systems: parts of phase space become completely uncorrelated after a finite time. E.g.: roulette wheel, dice.
11.7) Poincaré Recurrence Theorem

For a conservative system with a bounded phase space, a trajectory must return arbitrarily close to its initial position in a finite time.

Proof: Consider a finite part $S^2$ of phase space. The tube of trajectories emanating from $S^2$ must self-intersect otherwise the phase space volume would be infinite. However, because trajectories cannot intersect, the only possibility is that the phase space tube connects to its origin.

11.8) The logistic map

$$x_{n+1} = a \cdot x_n (1 - x_n) = F_a(x_n)$$

Feigenbaum 1970

$$0 \leq x_n \leq 1$$

$F(\frac{a}{2}) = \frac{a}{2}$

Fixed points:

$$x = a \cdot x (1 - x)$$

$$x = 0 \quad x = 1 - \frac{1}{a}$$
We take a > 1 s that both fixed points are in [0, 1) and a < 4 so that the map does not go outside [0, 1).

\[ a < 1 \]

**Stability of fixed points**

\[
\frac{dF_a}{dx} = a - 2ax
\]

\[
\frac{dF_a}{dx} \bigg|_{x=0} = a \quad \text{unstable if } a > 1
\]

\[
\frac{dF_a}{dx} \bigg|_{x=\frac{1}{a}} = a - 2a \left(1 - \frac{1}{a}\right) = -a + 2
\]

\[
\frac{dF_a}{dx} \bigg|_{x=\frac{1}{a}} \quad \text{stable } 1 < a < 3 \quad \text{unstable } a > 3
\]

What happens for \( a > 3 \)?

We can get periodic orbits.
a period 2 orbit is obtained when the map becomes unstable, i.e. for $a = 2$. For $a < 3$ the trajectory would have converged to the fixed point.

Let us look for a period 2 fixed point, i.e. a solution of $F_2^2(x) = x$

First, $F_2^2 = a(c_2 a x (1- x) - x (ax (1-x)))$

$F_2^2(x) = x \Rightarrow x = a^2 (1-x) (1 - ax (1-x))$

$x = \frac{1}{2}$ is a fixed point and solves this equation.

$a^2 (1-x) (1 - ax (1-x)) - 1 = (x - (1 - \frac{1}{2a})) P_2(x)$

$P_2(x) = (-ax^2 + a^2 (1+a)x - a(1+a))$
discriminant

\[ D = a^4 (a+1) (a-3) \]

1 real solution for \( a > 3 \)

a periodic 2 orbit appears at \( a = 3 \)

This is called period doubling. We have 2 period 2 fixed points on the same trajectory.

When \( F \) becomes unstable, a second period doubling occurs. We have 4 period 4 fixed points.

![Graph showing fixed points and a bifurcation diagram]

\[ \lim_{n \to \infty} a_n = 3.5699 \]

Feigenbaum number

\[ \delta_n = \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}} \]

\[ \lim_{n \to \infty} \delta_n = 4.669201609 \]

Lyapunov exponent:

\[ \lim_{N \to \infty} \frac{1}{N} \left( F_n(x_0 + \varepsilon) - F_n(x_0) \right) = \lambda \]

\( \lambda = 0 \) at bifurcation points, because the map becomes unstable.
Chaotic Logistic map

\( a < a_\infty \): no chaos, convergence to limit cycle

Lyapunov exponent

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \log |F_a'(x_i)| = \lambda(x_0)
\]

\( \lambda(x_0) = \lim_{N \to \infty} \frac{1}{N} \log |dF_a(x_0)| \)

\[
\frac{d}{dx} F_a^N(x_0) = F_a'(x_N) F_a'(x_{N-1}) \ldots F_a'(x_0)
\]

\( \lambda(x_0) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \log F'(x_i) \)

at bifurcation points \( F'(x_i) = 1 \)

\( \lambda(x_0)_{\text{bif}} = 0 \)

at other \( \lambda(x_0) < 0 \)

\( a_\infty < a < 4 \): limit cycles with all possible periods, e.g., 3

- Chaotic nonperiodic orbits with \( \lambda(x_0) > 0 \)

\[
\lambda(x_0)
\]

\( 0 \quad 3 \quad a_\infty \quad a \)
Canonical perturbation theory for 1dG

\[ H(\xi) = H_0(\xi) + \epsilon H_1(\xi) \]

\[ \xi = (x, p) \uparrow \]

completely integrable
can be rewritten in terms of
action angle variables

\[ K_0(\mathbf{Z}) = H_0(\xi(\phi_0, \mathbf{Z})) \uparrow \]

canonical transformation

We also rewrite the full Hamiltonian in terms of these coordinates

\[ K(\phi_0, \mathbf{Z}_0) = H(\xi(\phi_0, \mathbf{Z}_0)) = K_0(\mathbf{Z}_0) + \epsilon H_1(\xi(\phi_0, \mathbf{Z}_0)) \]

We assume that \( H(\xi) \) is also completely integrable.

\[ = \]

variables (\( \phi, \mathbf{Z} \)) so that \( H(\xi) = K(\mathbf{Z}) \)

We are going to determine the generating function for this transformation perturbatively

\( (\phi, \mathbf{Z}) \rightarrow (\phi_1, \mathbf{Z}_1) \rightarrow (\phi_2, \mathbf{Z}_2) \rightarrow \ldots \)

Generating function \( S(\phi_0, \mathbf{Z}) = \phi_0 + \epsilon S_1(\phi_0, \mathbf{Z}) \)

\[ + \epsilon^2 S_2(\phi_0, \mathbf{Z}) + \ldots \]

To \( \phi(\xi) \) : \[ \phi_0 = \frac{\partial S}{\partial \phi_0} = \phi_0 + \epsilon \frac{\partial S_1}{\partial \phi_0} \]

\[ \phi = \frac{\partial S}{\partial \xi} = \phi_0 + \epsilon \frac{\partial S_1}{\partial \phi_0} \]

\[ H(\mathbf{Z}) = H_0(\mathbf{Z}) + \epsilon H_1(\mathbf{Z}) \]

but also \( K(\phi_0, \mathbf{Z}_0) = K_0(\mathbf{Z}_0) + \epsilon K_1(\phi_0, \mathbf{Z}_0) \)

\[ K_0(\mathbf{Z}) = K_0(\mathbf{Z}_0) + \epsilon^2 K_1(\phi_0, \mathbf{Z}_0) + \ldots \]
\[ K(\phi_0, \gamma) = K_0(\gamma) + \varepsilon (K_1(\phi_0, \gamma) + \varepsilon \frac{\partial S_I}{\partial \phi_0} \frac{\partial \phi_0}{\partial \gamma}) \]

\[ \rightarrow H(\gamma) = K_1(\phi_0, \gamma) + \frac{\partial S_I}{\partial \phi_0} \frac{\partial \phi_0}{\partial \gamma} \]

\[ \text{Closed contour} \]

begin and end point of closed loop are the same

\[ \Rightarrow \gamma \text{ is the same at begin and end point} \]

but \( \gamma_0 = \gamma + \varepsilon \frac{\partial S_I}{\partial \phi_0} \Rightarrow \frac{\partial S_I}{\partial \phi_0} \text{ is periodic at fixed} \gamma \]

\( S_I \) should be such that \( \phi \) is an angle.

It should change by \( 2\pi \) about a closed contour \( \Rightarrow \) because \( \phi = \phi_0 + \varepsilon \frac{\partial S_I}{\partial \gamma} \)

We have that \( \frac{\partial S_I}{\partial \gamma} \) is periodic at fixed \( \gamma \)

We find that \( H_1(\gamma) = \langle K_1 \rangle \) because \( \frac{\partial \phi_0 \partial S_I}{\partial \phi_0} = 0 \)

and \( \frac{\partial S_I}{\partial \phi_0} = \frac{\langle K_1 \rangle - K_1(\phi_0, \gamma)}{\partial \gamma} \)

\[ S_I(\phi_0, \gamma) = S_I(\phi_0, \gamma) + f(\gamma) \]

solutions redefine zero of \( \phi \).
\[ H(\varepsilon) = H_0(\varepsilon) + \varepsilon H_1(\varepsilon) \quad \varepsilon = (q_1, \ldots, q_n, p_1, \ldots, p_n) \]

action angle variables \( (\phi_1, \ldots, \phi_n, \theta_1, \ldots, \theta_n) \)

\[ H_0 = \mathcal{K}_0 (\theta_1, \ldots, \theta_n) \]

\[ H_1 (\theta_1, \ldots, \theta_n) = \mathcal{K}_1 (\phi_1, \ldots, \phi_n, \theta_1, \ldots, \theta_n) + \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial \mathcal{K}_1}{\partial \phi_k} \frac{\partial \phi_l}{\partial \phi_k} \]

(Came from for \( \lambda \) dof)

\[ H (\mathcal{L}^{xy}) = \mathcal{K}_0 (\mathcal{L}^{xy}) + \varepsilon (\mathcal{K}_1 (\mathcal{L}^{xy})) \]

\[ = \mathcal{K}_0 (\mathcal{L}^{xy}) + (\mathcal{K}_1 - \mathcal{K}_0) \frac{\partial \mathcal{K}_0}{\partial \phi_0} + \varepsilon \mathcal{K}_1 (\mathcal{L}^{xy}) \]

generating function \( S = \sum \mathcal{K}_0 \mathcal{L}^{xy} + \varepsilon \mathcal{S}_1 (\mathcal{L}^{xy}, \mathcal{L}^{xy}) \)

\[ \Rightarrow H_0 (\mathcal{L}^{xy}) = \mathcal{K}_0 (\mathcal{L}^{xy}) \]

\[ H_1 (\mathcal{L}^{xy}) = \mathcal{K}_0 (\mathcal{L}^{xy}) + \sum_{k=1}^{n} \mathcal{K}_0 \frac{\partial \mathcal{S}_1}{\partial \phi_k} \]

\[ S_1 \text{ is periodic function of } \phi_0 \]

\[ \Rightarrow \int_0^{2\pi} \mathcal{K}_0 \text{d} \phi_0 \frac{\partial S_1}{\partial \phi_0} = 0 \]

\[ \Rightarrow \quad H (\mathcal{L}^{xy}) = \mathcal{K}_1 \]

\[ \sum_{k=1}^{n} \mathcal{K}_0 \frac{\partial \mathcal{S}_1}{\partial \phi_k} = \mathcal{K}_1 - \mathcal{K}_1 (\mathcal{L}^{xy}, \mathcal{L}^{xy}) \]

\[ S_1 \text{ is periodic} \Rightarrow \text{it can be Fourier decomposed} \]

\[ S_1 = \sum_{m} \mathcal{S}_1 (\mathcal{L}^{xy}, \text{\text{lin} } \phi_0) \]
\[ \sum_x \left( m_x \alpha_0 \right) S_x (\{ \omega_x \}, \{ m_y \}) = \frac{\langle \mathbf{k} \rangle - \mathbf{E}}{\mathbf{v}} \]

\[ \sum_{\mathbf{k}} \left( \mathbf{k} \alpha_0 \right) \left( \mathbf{E} \alpha_0 \right) \]

\[ x e^{i m_x \phi_0} \]

If the frequencies are commensurate, pt. diverges for some \( n_k \)

This happens if ratios of the frequencies are rational.

Then we have rational tori; the Poincaré section becomes discrete, i.e. we are at a fixed point of a power of the Poincaré map.
Under a perturbation the rational tori split into an equal number of alternating elliptic and hyperbolic fixed points.

We will illustrate this theorem for the kicked rotor

$2\varepsilon$:
\[ \phi_{n+1} = (\phi_n + J_{n+1}) \mod 2\pi \]
\[ J_{n+1} = J_n + \varepsilon \sin \phi_n \]

Consider $Z_0 = Z\varepsilon = 0$ and $J = \frac{2\pi}{\varepsilon} \frac{e^2}{2} \equiv J_F$

then $Z_0^k (J, \phi) = J, \phi$

applying the transformation

$k$ times adds an integer multiple of $2\pi$ to $\phi$

$Z_0^k (\phi, J_F) \rightarrow Z_0^k (\phi, J_F)$

$Z_0^k (\phi, J_F)$

$k \neq J_F$ angles minus

$k < J_F$ angles minus

$k > J_F$ angles minus

$k > J_F$ angles minus

$k < J_F$ angles minus

$k = J_F$ angles minus

$k = J_F$ angles minus

$k = J_F$ angles minus

What happens for $\varepsilon > 0$?

$Z_\varepsilon$ is an area preserving map

$\phi_{n+1} = (\phi_n + J_n + \varepsilon \sin \phi_n) \mod 2\pi$

$J_{n+1} = J_n + \varepsilon \sin \phi_n$

$\frac{\partial \phi_{n+1}}{\partial \phi_n} \frac{\partial J_{n+1}}{\partial J_n} = \frac{1 + \varepsilon \cos \phi_n}{\varepsilon \cos \phi_n} = 1$
\( e^c (\phi, y) \) is a continuous function for \( c = 1 \)

\( \Rightarrow \) it maps a circle to a deformed closed curve with the same area.

\[ \text{circle} \rightarrow \text{deformed curve} \]

For larger \( \phi \), the angles move faster and for smaller \( \phi \), they move slower. We can find a curve such that the \( \phi \) values are not changed by the map.

The intersection points are fixed points. Because the area is preserved, their number is necessarily even.

Let us look at an intersection:

- hyperbolic fixed point
- elliptic fixed point
If \((\phi_0, J_0)\) is a fixed point of \(Z_\varepsilon^{K}\) smaller than \(L\), then \((\phi_0, J_0) = Z_\varepsilon^m (\phi_0, J_0)\) in also a fixed point of \(Z_\varepsilon^{K}\). They all have period \(K\).

Assume they have a period \(M \neq 2\), \(M < K\), then \((\phi_0, J_0) = Z_\varepsilon^m (\phi_0, J_0)\)

\[
\Rightarrow Z_\varepsilon^S (\phi_0, J_0) = Z_\varepsilon^m Z_\varepsilon^S (\phi_0, J_0)
\]

\[
\Rightarrow Z_\varepsilon^m (\phi_0, J_0) = (\phi_0, J_0)
\]

\[
\Rightarrow \text{period would be less than } L \Rightarrow m = 0
\]

\[
\Rightarrow \text{fixed points occur in multiples of } K.
\]

An elliptic fixed point cannot be mapped into a hyperbolic fixed point by continuity. 

\(\Rightarrow\) Elliptic and hyperbolic fixed points.

12.1) **Homoclinic tangency**

- Stable manifold: \(\lim_{\varepsilon \to 0} Z_\varepsilon^K (x) = x_0\)
- Unstable manifold: \(\lim_{\varepsilon \to 0} Z_\varepsilon^K (x_0) = y_0\)

- Hyperbolic fixed point

- Stable and unstable manifolds cannot intersect elliptic orbits.
point on an elliptic orbit stays on the ellipse which contradicts convergence to $p_0$.

We and $w_1$ cannot self intersect.

Then two points would be mapped to $p_0$ which would violate invertability.

$w_s (or w_u)$ of different fixed points cannot intersect cannot go to 3 points $p_0, p_1, p_2$.

However, the stable and unstable manifolds can intersect. This leads to the homoclinic tangle.

Homoclinic point $w_s$ and $w_u$ of the same fixed point intersect.

Heteroclinic point $w_s$ and $w_u$ of different fixed points intersect.

$x_1 = Z_\varepsilon (x_0) \quad x_{i+1} = Z_n \circ x_i \quad x_u = Z_\varepsilon (x_0)$

has to stay both on $w_2$ and $w_1$.  

for $d < 0$ there are no many intersections. This leads to the homoclinic tangle and the transition to chaos.
12. (The KAM Theorem)

All rational tori are unstable. This is not the case for irrational tori.
Instability occurs for \((2, \tilde{m}) = 0\) for \(\epsilon \ll m, y\).
So we have to stay away from these frequencies
such that
\[
\sum_{m \neq 0} \frac{\omega(\tilde{m})}{i (\tilde{m}, \tilde{r}_0)} e^{i \omega(\tilde{m}) t}
\]
converges for \(L \to \infty\).

Exclude frequencies with
\(1 | \tilde{m}, \tilde{r}_0 | \tilde{r}_0\), but we also
will end up with a finite part of phase space.

Consider first the case that \(H_0 = J_1 + \lambda J_2\)
\(\Rightarrow \lambda = 1, \nu = 2\)

Small denominator relation \(m_1 + 2 m_2 = 0\)
We cannot adjust \(J_2\) to stay away from a
small denominator.
This leads to the condition that \(\nu_2(\tilde{x})\) is invertible
\(\det \partial^2 \nu_2 \tilde{x} \neq 0\).\(\partial \tilde{x}\) and \(\partial^2 \tilde{x}\) are both
This is called the Hessian condition
\[
\begin{array}{c|c}
\nu_2(\tilde{x}) & \text{no good} \\
\tilde{x} & \text{good}
\end{array}
\]
The Hessian condition made sure that the frequencies vary if $J_2$ is varied.

(ii) We need to stay away sufficiently far from resonances. This leads to the Diophantine condition

$$\exists \theta \in [1, J_2] \text{ satisfying the Hessian condition}$$

$\theta \neq \text{dof}$ because otherwise all of phase space will be excluded.

(Volume of integers $< R \cap R^{\# \text{dof}}$)

$$\forall \theta \in [1, J_2] \sum_{\text{dof}} \frac{\psi_{\text{int}}(\theta)}{i \Delta \omega \Delta x}$$

Converges for a finite part of phase space.

$\theta = \{ \theta \mid \text{frequencies are outside excluded bands} \}$

Choose $L$ such that

$$\sum_{\text{dof}} \frac{\psi_{\text{int}}(\theta)}{i \Delta \omega \Delta x} e^{i \phi} = O(\epsilon)$$

Choose $\epsilon \ll 1$ such that for $\sum \frac{\psi_{\text{int}}(\theta)}{i \Delta \omega \Delta x} e^{i \phi}$ converges (finite series).

Use a canonical transformation to bring $H_1$ to action-angle variables $\theta \chi(\theta)$.

(Note that $\sum \frac{\psi_{\text{int}}(\theta)}{i \Delta \omega \Delta x} e^{i \phi} = O(\epsilon)$.)
\[ H_{01} = H + \varepsilon_1 H_0 \]

\[ H = H_{01} + \varepsilon_1 H_2 \]

Next we bring \( \varepsilon_1 H_2 \) to action angle variables to \( O(\varepsilon^2) \).

Choose \( L \) such that

\[ \sum_{|\hat{m}|<L} \frac{K_0(\hat{m})}{(\hat{m}, \hat{m})} \sim O(\varepsilon) \]

Choose \( R \) such that

\[ \sum_{|\hat{m}|>R} \frac{K_0(\hat{m})}{(\hat{m}, \hat{m})} < \frac{1}{R} \]

Then

\[ \sum_{|\hat{m}|<L} \frac{K_0(\hat{m})}{(\hat{m}, \hat{m})} \text{ converges} \]

\[ H_{02} = H_{01} + \varepsilon_1 H_1 \]

\[ H = H_{02} + \varepsilon_1 H_2 \]

Convergence proceeds geometrically

\[ R = \frac{1}{\varepsilon_1} \]

proving the UAM theorem amounts to proving that the measure of \( \Sigma_0 \) is finite.