I.d) Lagrangian for a free particle

$$\vec{r} = \vec{r}' + \vec{v}_f t \quad \Rightarrow \quad \vec{v} = \vec{v}' + \vec{v}_f$$

Lagrangian should be invariant under a Galilei transformation

$$\Rightarrow \text{if } L(v^2) = \sum_k a_k (v^2)^k$$

then
$$L(v')^2 = \sum_k a_k (v'^2)^k$$
$$= \sum_k (\vec{v} - \vec{v}_f)^2 a_k$$
$$= \sum_k \Big(\vec{v}^2 + \underbrace{v_f^2 - 2\vec{v}\cdot\vec{v}_f}_{\frac{d}{dt}(v_f^2 t - 2\vec{v}_f\cdot\vec{r})}\Big)^k a_k$$

only total derivative for $k = 1$

$$\Rightarrow \quad L = \underset{\text{convention}}{\frac{1}{2}} \underset{\text{mass}}{m} \vec{v}^2$$

system of free particles $\quad L = \sum_a \frac{1}{2} m_a \vec{v}_a^2$