Ve Perturbation theory for anharmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 + \frac{1}{2} \varepsilon x^4$$

$$\Rightarrow \ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0$$

perturbative expansion $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$

$$\Rightarrow \ddot{x}_0(t) + \varepsilon \ddot{x}_1(t) + \varepsilon^2 \ddot{x}_2(t) + \omega_0^2 x_0(t) + \omega_0^2 \varepsilon x_1(t) + \omega_0^2 \varepsilon^2 x_2(t)$$
$$+ \varepsilon x_0^3(t) + 3 \varepsilon^2 x_0^2 x_1(t) + \mathcal{O}(\varepsilon^3) = 0$$

$\mathcal{O}(\varepsilon^0)$; $\ddot{x}_0 = -\omega_0^2 x_0$

$\mathcal{O}(\varepsilon^1)$; $\ddot{x}_1 + \omega_0^2 x_1 + x_0^3 = 0$

initial conditions $\left.\begin{array}{l} x(0) = a \\ \dot{x}(0) = 0 \end{array}\right\} \Rightarrow \forall k > 0 \; \begin{array}{l} x_k(0) = 0 \\ \dot{x}_k(0) = 0 \end{array}$

$\left.\begin{array}{l} x_0(0) = a \\ \dot{x}_0(0) = 0 \end{array}\right\} \Rightarrow x_0 = a \cos \omega_0 t$

$$\Rightarrow \ddot{x}_1 + \omega_0^2 x_1 = a^3 \cos^3 \omega_0 t = -\frac{3}{4} a^3 \cos \omega_0 t - \frac{1}{4} a^3 \cos \omega_0 3t$$

$\Rightarrow$ parametric resonance $\Rightarrow$ amplitude will diverge for $t \to \infty$

diverges for $t \to \infty$

Solution $x_1(t) = \frac{-a^3}{8\omega_0^2} \left(3\omega_0 t \sin \omega_0 t + \frac{1}{4} \cos \omega_0 t - \frac{1}{4} \cos 3\omega_0 t \right)$

reason for divergence true frequency $\omega = \omega_0 + \delta\omega$

expand $\cos((\omega_0 + \delta\omega) t) = \cos \omega_0 t - \delta\omega \, t \sin \omega_0 t$

diverges