Vb. Parametric Resonance

We consider a system with a time-dependent mass and spring constant:

\[ \frac{d}{dt} (m(t) \dot{x}) + c(t) x = 0 \]

\[ = m(t) \ddot{x} + c(t) \dot{x} + k(t) x = 0 \]

Define \[ \tau = \int \frac{dt}{m(t)} \]

\[ \Rightarrow \frac{d}{d\tau} x = \frac{dx}{dt} \frac{dt}{d\tau} m(t) \]

\[ \frac{d^2 x}{d\tau} = \frac{d^2 x}{dt^2} \frac{dt}{d\tau} m(t) + \frac{dx}{dt} \frac{dt}{d\tau} m(t) \]

\[ = \frac{d^2 x}{d\tau^2} + m(t) \dot{x} x = 0 \]

This means that the most general problem is given by:

\[ \frac{d^2 x}{d\tau^2} + \omega^2(t) x = 0 \]

Assumption: \( \omega(t) \) is periodic with period \( T \)

\[ \omega(t+T) = \omega(t) \]

Solution:

\[ \begin{pmatrix} x_1(t+T) \\ x_2(t+T) \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \]

We can always choose a basis where this matrix is diagonal:

\[ x_1(t+T) = \alpha_1 x_1(t) \]

\[ x_2(t+T) = \beta_1 x_2(t) \]