III c) Motion in a central field

$U = U(|\vec{r}_1 - \vec{r}_2|)$ problem can be reduced to relative coordinate $\vec{r} = |\vec{r}_1 - \vec{r}_2|$

Lagrangian in spherical coordinates

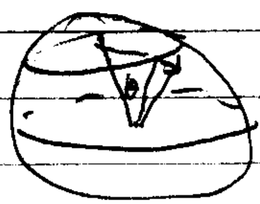
$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) - U(r)$$

L is invariant for rotations about $r=0$

then $\theta \to \theta + \delta\theta$ $\quad \dot{\theta} \to \dot{\theta}$

$\varphi \to \varphi + \delta\varphi$ $\quad \dot{\varphi} \to \dot{\varphi} - \frac{\cos\theta}{\sin\theta}\dot{\varphi}\,\delta\theta$

$r \to r$



$r\sin\theta\dot{\varphi} = r\sin(\theta+\delta\theta)(\dot{\varphi}+\delta\dot{\varphi})$

$\Rightarrow \quad 0 = r\dot{\varphi}\cos\theta\,\delta\theta + r\sin\theta\,\delta\dot{\varphi}$

$$\Rightarrow \delta L = r^2(\dot{\phi}+\delta\dot{\phi})^2 \sin^2(\theta+\delta\theta) - r^2\dot{\phi}^2\sin^2\theta$$

$$= 2r^2\,\underset{\underset{-\frac{\cos\theta}{\sin\theta}\dot{\varphi}\delta\theta}{\parallel}}{\delta\dot{\phi}}\,\sin^2\theta + r^2\dot{\phi}^2\, 2\sin\theta\cos\theta\,\delta\theta$$

$\Rightarrow \delta L = 0 \quad \Rightarrow \vec{L} = \vec{r} \times \vec{p}$ is conserved

$\Rightarrow \quad \vec{r} \perp L \quad \Rightarrow$ motion is in the plane

choose $\theta = 90°$ for this plane

$$\Rightarrow \quad L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$