Example $x_1^2 + x_2^2 = R^2 \Rightarrow \vec{\nabla} f = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$

If we have k constraints, then constraint forces

$$\vec{C}_i = \sum_{\ell=1}^{k} \underbrace{\lambda_\ell}_{\text{unknowns}} \vec{\nabla}_i f_\ell$$

$\vec{\nabla}_i = \frac{\partial}{\partial \vec{x}_i}$ (constraints $f_\ell(x_1, \dots, x_N, t) = 0$, $\ell = 1, \dots, k$)

$\Rightarrow$ $N + K$ unknowns

NE + constraints $N + K$ equations

IIj) Lagrange Eqs with constraints

NE with constraints $m_i \ddot{\vec{x}}_i + \vec{\nabla}_i V - \sum \lambda_\ell \vec{\nabla}_i F_\ell = 0$

constraints

$$\vec{\nabla}_i = \frac{\partial q_K}{\partial x_i} \frac{\partial}{\partial q_K} \text{ or } (\vec{\nabla}_i) \cdot \frac{\partial \vec{x}_i}{\partial q_K} = \frac{\partial}{\partial q_K}$$

generalized coordinates

So to get Newton's law in generalized coordinates, we multiply by $\cdot \frac{\partial \vec{x}_i}{\partial q_K}$

Let us look at $\ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q_K}$

$$\ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q_K} = \frac{d}{dt}\left[\dot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q_K}\right] - \dot{\vec{x}}_i \cdot \frac{d}{dt} \frac{\partial x_i}{\partial q_K}$$

$$\frac{\partial \dot{\vec{x}}_i}{\partial \dot{q}_K} \Leftarrow \quad \dot{\vec{x}}_i = \frac{\partial \vec{x}_i}{\partial q_K} \dot{q}_K + \frac{\partial x_i}{\partial t} \quad \Rightarrow \frac{\partial \dot{\vec{x}}_i}{\partial \dot{q}_K} = \frac{\partial \vec{x}_i}{\partial q_K}$$