The $2 \phi_0$ term just redefines $\phi$ and can be ignored =) $S_1$ is periodic in $\phi_0$.

$$= \delta \frac{\partial S_1}{\partial \phi_0} = 0$$

$$= \frac{1}{2m} \int d\phi_0 \kappa_1(\phi_0, 2)$$

So we find

$$\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \kappa_1 \rangle - \kappa_1(\phi_0, 2)}{\nu(2)}$$

$S_1$ can be obtained by integration w.r.t. $\phi_0$

$$= S_1 = F(\phi_0, 2) + \sigma(2)$$

$q_0 = \phi + \frac{3}{2} \frac{\partial S_1}{\partial \phi_0}$

$$\phi = \phi_0 + \frac{3}{2} S_1 + \frac{3}{2} \sigma$$

$\text{Constant of motion, redefines } \phi$

ignore.