Homework Set # 11, due December 3, 2007

1. Solve the harmonic oscillator in one dimension using action-angle variables.

2. The aim of this problem is to illustrate that Liouville's theorem is valid for a Hamiltonian for which the energy is not conserved. We consider the damped harmonic oscillator with equation of motion

$$m\frac{d^2x}{dt^2} = \alpha\frac{dx}{dt} + kx = 0. \tag{1}$$

   (a) Show the the Lagrangian is given by

$$L = [\frac{m}{2}(\frac{dx}{dt})^2 - \frac{k}{2}x^2]e^{\frac{\alpha}{m}t}. \tag{2}$$

   (b) Calculate the canonical momentum and obtain the Hamiltonian.

   (c) Write down the Hamiltonian equations of motion.

   (d) Find the solution for initial consitions $(x(0), p(0))$. Do this by first solving (1) and then solve the Hamilton equation for dp/dt.

   (e) Write the solution as

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = A \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}. \tag{3}$$

   Show that $\det A = 1$. This shows that the phase space volume does not change even though the coordinate and velocity approach zero for large times. This is not a contradiction because the momentum is enhanced by a factor $\exp(t\alpha/m)$ with respect to the velocity.

3. In this problem we study the cat map numerically. Consider a square $[0,1] \times [0,1]$ and discretize it with 50 points in the x direction and y direction. Take your favorite figure and discritize it by putting black points on the lattice. For example, you could use black squares with sides equal to the lattice spacing or circles with diameter equal to the lattice spacing. Apply the cat map

$$x_{n+1} = (2x_n + y_n) \bmod 1, \tag{4}$$
$$y_{n+1} = (x_n + y_n) \bmod 1, \tag{5}$$

   to black points.

   (a) Make a graph of the orginal figure, the figure after one iteration (i.e. all points have been updated once), and the figure after 10 iterations.

   (b) Determine after how many iterations the original figure is restored.

   If this takes too much computer time, you can also work with a smaller lattice, for example a 10×10 lattice.