III - AN INTRODUCTION TO CLASSICAL CHAOTIC MOTION

The aim of this section is to introduce the concept of chaos in classical mechanics of conservative systems.

Giving a rigorous presentation of the material would require a highly technical language, and elaborate mathematical tools. Our purpose is rather to present an elementary intuitive approach to the subject, in order to get a physical insight into the main ideas. The material of this section is far from being exhaustive, and many fundamental questions have been omitted (such as perturbation theory, bifurcating orbits, mechanism of destruction of tori, dissipative systems, etc.).

In what follows, we limit ourselves to the study of classical Hamiltonian systems which are conservative and time-reversal invariant; moreover, we consider only initial conditions for which the motion can take place only in a bounded region of the phase space.

III.1 FROM REGULAR TO CHAOTIC MOTION

All conservative Hamiltonian systems with \( N \) degrees of freedom have in common three essential properties:

i) for a given set of initial conditions, the dimensionality of the accessible surface in phase space is less or equal to \( 2N-1 \); since the system is conservative, the energy is constant along this "energy surface".

ii) From Liouville's theorem, we know that the volume element in phase space is conserved. In other words, the Hamiltonian flow, which preserves the measure in phase space, is incompressible.

iii) Trajectories in phase space cannot cross.

Apart from these features which are shared by all systems, the motion in phase space can exhibit a great variety of behaviours. For instance, one may ask how does a given volume element evolve with time: does it tend to cover the whole energy surface \( S_E \) as time goes to infinity or does it remain in a restricted part of \( S_E \)? Does it conserve approximately its initial shape, or does it display more or less dramatic deformations with time? According to the answers to such questions, one can define a hierarchy of regularity for dynamical systems. As we shall see now, the most regular systems, lying at the bottom of this classification, can be used as clocks, whereas, at the opposite side, the most chaotic systems (*) can be used as random number generators.

(*) We speak here of dynamical systems in the enlarged sense of area-preserving mappings (see Sect. III.2).