the system are evidently unchanged by a displacement along the axis of a co-ordinate which does not appear in the potential energy, and so the corresponding component of the momentum is conserved. For example, in a uniform field in the z-direction, the x and y components of momentum are conserved.

The equation (7.1) has a simple physical meaning. The derivative $\partial L/\partial \mathbf{r}_a = -\partial U/\partial \mathbf{r}_a$ is the force $\mathbf{F}_a$ acting on the ath particle. Thus equation (7.1) signifies that the sum of the forces on all the particles in a closed system is zero:

$$\sum_a \mathbf{F}_a = 0. \tag{7.4}$$

In particular, for a system of only two particles, $\mathbf{F}_1+\mathbf{F}_2 = 0$: the force exerted by the first particle on the second is equal in magnitude, and opposite in direction, to that exerted by the second particle on the first. This is the equality of action and reaction (*Newton's third law*).

If the motion is described by generalised co-ordinates q_i, the derivatives of the Lagrangian with respect to the generalised velocities

$$p_i = \partial L/\partial \dot{q}_i \tag{7.5}$$

are called *generalised momenta*, and its derivatives with respect to the generalised co-ordinates

$$F_i = \partial L/\partial q_i \tag{7.6}$$

are called *generalised forces*. In this notation, Lagrange's equations are

$$\dot{p}_i = F_i. \tag{7.7}$$

In Cartesian co-ordinates the generalised momenta are the components of the vectors $\mathbf{p}_a$. In general, however, the p_i are linear homogeneous functions of the generalised velocities $\dot{q}_i$, and do not reduce to products of mass and velocity.

PROBLEM

A particle of mass m moving with velocity $\mathbf{v}_1$ leaves a half-space in which its potential energy is a constant U_1 and enters another in which its potential energy is a different constant U_2. Determine the change in the direction of motion of the particle.

SOLUTION. The potential energy is independent of the co-ordinates whose axes are parallel to the plane separating the half-spaces. The component of momentum in that plane is therefore conserved. Denoting by θ_1 and θ_2 the angles between the normal to the plane and the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the particle before and after passing the plane, we have $v_1 \sin\theta_1 = v_2 \sin\theta_2$. The relation between v_1 and v_2 is given by the law of conservation of energy, and the result is

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\left[1+\frac{2}{mv_1^2}(U_1-U_2)\right]}.$$

§8. Centre of mass

The momentum of a closed mechanical system has different values in different (inertial) frames of reference. If a frame K' moves with velocity $\mathbf{V}$ relative to another frame K, then the velocities $\mathbf{v}_a'$ and $\mathbf{v}_a$ of the particles relative to the two frames are such that $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$. The momenta $\mathbf{P}$ and $\mathbf{P}'$ in the two frames are therefore related by

$$\mathbf{P} = \sum_a m_a \mathbf{v}_a = \sum_a m_a \mathbf{v}_a' + \mathbf{V} \sum_a m_a,$$

or

$$\mathbf{P} = \mathbf{P}' + \mathbf{V} \sum_a m_a. \tag{8.1}$$

In particular, there is always a frame of reference K' in which the total momentum is zero. Putting $\mathbf{P}' = 0$ in (8.1), we find the velocity of this frame:

$$\mathbf{V} = \mathbf{P}/\sum m_a = \sum m_a \mathbf{v}_a/\sum m_a. \tag{8.2}$$

If the total momentum of a mechanical system in a given frame of reference is zero, it is said to be *at rest* relative to that frame. This is a natural generalisation of the term as applied to a particle. Similarly, the velocity $\mathbf{V}$ given by (8.2) is the velocity of the "motion as a whole" of a mechanical system whose momentum is not zero. Thus we see that the law of conservation of momentum makes possible a natural definition of rest and velocity, as applied to a mechanical system as a whole.

Formula (8.2) shows that the relation between the momentum $\mathbf{P}$ and the velocity $\mathbf{V}$ of the system is the same as that between the momentum and velocity of a single particle of mass $\mu = \Sigma m_a$, the sum of the masses of the particles in the system. This result can be regarded as expressing the *additivity of mass*.

The right-hand side of formula (8.2) can be written as the total time derivative of the expression

$$\mathbf{R} \equiv \sum m_a \mathbf{r}_a/\sum m_a. \tag{8.3}$$

We can say that the velocity of the system as a whole is the rate of motion in space of the point whose radius vector is (8.3). This point is called the *centre of mass* of the system.

The law of conservation of momentum for a closed system can be formulated as stating that the centre of mass of the system moves uniformly in a straight line. In this form it generalises the law of inertia derived in §3 for a single free particle, whose "centre of mass" coincides with the particle itself.

In considering the mechanical properties of a closed system it is natural to use a frame of reference in which the centre of mass is at rest. This eliminates a uniform rectilinear motion of the system as a whole, but such motion is of no interest.

The energy of a mechanical system which is at rest as a whole is usually called its *internal energy* E_i. This includes the kinetic energy of the relative motion of the particles in the system and the potential energy of their interaction. The total energy of a system moving as a whole with velocity V can be written

$$E = \tfrac{1}{2}\mu V^2 + E_i. \tag{8.4}$$