Mathematically, the equations (2.6) constitute a set of s second-order equations for s unknown functions $q_i(t)$. The general solution contains $2s$ arbitrary constants. To determine these constants and thereby to define uniquely the motion of the system, it is necessary to know the initial conditions which specify the state of the system at some given instant, for example the initial values of all the co-ordinates and velocities.

Let a mechanical system consist of two parts A and B which would, if closed, have Lagrangians L_A and L_B respectively. Then, in the limit where the distance between the parts becomes so large that the interaction between them may be neglected, the Lagrangian of the whole system tends to the value

$$\lim L = L_A + L_B. \tag{2.7}$$

This additivity of the Lagrangian expresses the fact that the equations of motion of either of the two non-interacting parts cannot involve quantities pertaining to the other part.

It is evident that the multiplication of the Lagrangian of a mechanical system by an arbitrary constant has no effect on the equations of motion. From this, it might seem, the following important property of arbitrariness can be deduced: the Lagrangians of different isolated mechanical systems may be multiplied by different arbitrary constants. The additive property, however, removes this indefiniteness, since it admits only the simultaneous multiplication of the Lagrangians of all the systems by the same constant. This corresponds to the natural arbitrariness in the choice of the unit of measurement of the Lagrangian, a matter to which we shall return in §4.

One further general remark should be made. Let us consider two functions $L'(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$, differing by the total derivative with respect to time of some function $f(q, t)$ of co-ordinates and time:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t). \tag{2.8}$$

The integrals (2.1) calculated from these two functions are such that

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\,dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt + \int_{t_1}^{t_2} \frac{df}{dt}\,dt = S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1),$$

i.e. they differ by a quantity which gives zero on variation, so that the conditions $\delta S' = 0$ and $\delta S = 0$ are equivalent, and the form of the equations of motion is unchanged. Thus the Lagrangian is defined only to within an additive total time derivative of any function of co-ordinates and time.

§3. Galileo's relativity principle

In order to consider mechanical phenomena it is necessary to choose a *frame of reference*. The laws of motion are in general different in form for different frames of reference. When an arbitrary frame of reference is chosen, it may happen that the laws governing even very simple phenomena become very complex. The problem naturally arises of finding a frame of reference in which the laws of mechanics take their simplest form.

If we were to choose an arbitrary frame of reference, space would be inhomogeneous and anisotropic. This means that, even if a body interacted with no other bodies, its various positions in space and its different orientations would not be mechanically equivalent. The same would in general be true of time, which would likewise be inhomogeneous; that is, different instants would not be equivalent. Such properties of space and time would evidently complicate the description of mechanical phenomena. For example, a free body (i.e. one subject to no external action) could not remain at rest: if its velocity were zero at some instant, it would begin to move in some direction at the next instant.

It is found, however, that a frame of reference can always be chosen in which space is homogeneous and isotropic and time is homogeneous. This is called an *inertial frame*. In particular, in such a frame a free body which is at rest at some instant remains always at rest.

We can now draw some immediate inferences concerning the form of the Lagrangian of a particle, moving freely, in an inertial frame of reference. The homogeneity of space and time implies that the Lagrangian cannot contain explicitly either the radius vector $\mathbf{r}$ of the particle or the time t, i.e. L must be a function of the velocity $\mathbf{v}$ only. Since space is isotropic, the Lagrangian must also be independent of the direction of $\mathbf{v}$, and is therefore a function only of its magnitude, i.e. of $\mathbf{v}^2 = v^2$:

$$L = L(v^2). \tag{3.1}$$

Since the Lagrangian is independent of $\mathbf{r}$, we have $\partial L/\partial \mathbf{r} = 0$, and so Lagrange's equation is†

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = 0,$$

whence $\partial L/\partial \mathbf{v}$ = constant. Since $\partial L/\partial \mathbf{v}$ is a function of the velocity only, it follows that

$$\mathbf{v} = \text{constant}. \tag{3.2}$$

Thus we conclude that, in an inertial frame, any free motion takes place with a velocity which is constant in both magnitude and direction. This is the *law of inertia*.

If we consider, besides the inertial frame, another frame moving uniformly in a straight line relative to the inertial frame, then the laws of free motion in

† The derivative of a scalar quantity with respect to a vector is defined as the vector whose components are equal to the derivatives of the scalar with respect to the corresponding components of the vector.